MA 266 - LECTURE NOTES

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Sec. 1.1-1.3 - Modeling via Diff Eq/Direction fields/some solutions/definitions

- What is a differential equation? What is a solution to a differential equation? (hard concept)
  - **Algebraic equation:**
    - $x^2 - 1 = 0$ or $x^2 + 1 = 0$
    - Questions? what are solutions? Are there even any solution? If so how many?
      - We can always check a number is a solution: Is $x = 1$, a solution of $x^2 - 1 =$ since $1^1 - 1 = 0$.
  - **Differential equations:** equation that have derivatives of functions in them
    - $\frac{dy}{dt} = 2y$ or $\frac{dy}{dt} = 4y + e^t$
    - What are solutions to differential equations? They are functions! Tricky because we also have to worry about domains.
    - Is there even a solution? (Existence) If so, how many? (Uniqueness?)
    - We can always check if a function is a solution to a differential equation
    - **Example:** Show $y = 9e^{2t}$ is a solution to $\frac{dy}{dt} = 2y$.
      - **Solution:**

$$LHS \stackrel{?}{=} RHS$$
$$\frac{d}{dt}\left(9e^{2t}\right) \stackrel{?}{=} 2 \cdot 9e^{2t}$$
$$18e^{2t} \stackrel{\checkmark}{=} 2 \cdot 9e^{2t}$$

      - What about $y = 9e^{2t} + 1$? Is this a solution? No!
    - **Check** that $y(t) = 1 + t$ is a solution to

$$\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t}.$$

Studying first order differential equations.

- What are diff eqs used for? This is class is about predicting the future.
  - **Example:** Meteorologists try to do it all the time with the weather, and they get it wrong all the time. This means modeling is hard.
  - **Example**: For example the solution to the equation

$$\frac{dP}{dt} = k \cdot P,$$

models the population $P(t)$ of a species at time t.

- The standard form of a **first order different equation** (meaning it has only first derivatives in the equation) is
$$\frac{dy}{dt} = f(t, y).$$
Recall that y is really a FUNCTION. Like $y = y(t)$.
  - t is the indepedent variable.
- An **initial value problem** (**IVP**) is a diff eq., with an initial condition:
$$\frac{dy}{dt} = f(t, y). \quad y(t_0) = y_0$$
  - **Example:**
$$\frac{dy}{dt} = 2y \;\; y(0) = 9$$
    * **Question:** Is $y(t) = 9e^{2t}$ a solution to this IVP?
      · Yes remember we already checked earlier that $y(t) = 9e^{2t}$ is a solution to the ODE and clearly $y(0) = 9e^{2\cdot 0} = 9$.
- A **particular solution** to an ODE is simply one of the functions $y = y(t)$ that satisfy a diff. eq. $y'(t) = f(t, y(t))$ for all t.
- A **General Solution** is one that includes all possible solutions to any IVP involving a specific ODE parametrized by parameters.
  - **Example:**
    * To find the general solution to $\frac{dy}{dt} = 2y$, we can separate the y's and t's to one side and then integrate
$$\begin{aligned} \frac{dy}{dt} = 2y &\iff \frac{dy}{y} = 2dt \\ &\iff \int \frac{dy}{y} = \int 2dt \\ &\iff \ln|y| = 2t + C \\ &\iff |y| = e^{2t+C} = Ke^{2t}, \text{ where } K = e^C \\ &\iff y = ce^{2t}, \text{ where } c = \pm K. \end{aligned}$$
    * Thus the general solution must be of the form
$$y = ce^{2t}.$$
    * There will be a whole section on this technique!
- An **equilibrium solution** $y(t) = y_0$ are the constant solutions of an ODE. That is
$$\frac{dy(t)}{dt} = 0 \text{ for all } t.$$

Example: Find the equilibrium solutions of the following equation

Suppose we have
$$y' = y^3 + y^2 - 6y$$
for what values of y is $y(t)$ in equilibrium, increasing, decreasing? Get
$$y' = y(y-2)(y+3),$$
and create a sign chart as we used to do in calculus. Equilibrium solutions are $y = -3, 0, 2.$ and get decreasing for $(-\infty, -3) \cup (0, 2)$ and increasing for $(-3, 0) \cup (2, \infty)$.

Solutions to some differential Equations.

- **A Linear Diff Eq.:** Pick your favorite real numbers a, b, y_0 and consider the IVP

$$\frac{dy}{dt} = ay - b, \quad y(0) = y_0$$

  - The **general solution** to this diff eq is

$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}$$

  - I will show you how one can gets this very soon!
- **Example:** Find the solution to

$$\frac{dy}{dt} = -2y + 8, \quad y(0) = 5,$$

  - **Solution:** Then $a = -2, b = -8$ and $y_0 = 9$ so the solution is

$$y(t) = 4 + \left(5 - \frac{8}{2}\right) e^{-2t} = 4 + e^{-2t}.$$

Studying general differential equations.

- We will only study **ordinary differential equations** (ODE): contains only ordinary derivatives: $\frac{d^2y}{dt^2} + \frac{dy}{dt} = -1$
- There is a whole separate course where one can study **partial differential equations(**PDE): $\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = -1$
- System of equations:

$$\frac{dx}{dt} = x - xy$$
$$\frac{dv}{dt} = y - 3x$$

- The **order** of the equation speaks to the highest derivative in the equation

$$y' + 3y = 0 \text{ 1st order}$$
$$y'' + 3y6\prime = 02t \text{ 2nd order}$$
$$\frac{d^5y}{dt} + \frac{dy}{dt} = y \text{ 5th order}$$
$$u_{xx} + u_{yy} = 0 \text{ 2nd order}$$

- An ODE is called linear if it is linear in y, it is of the form

$$a_n(t)y^{(n)} + a_{n-1}(y)y^{(n-1)} + \cdots + a_0(t)y = g(t)$$

  - **Linear:** $y' + 4y = 0$, $y'' + 3e^y y$
  - **Nonlinear:** $\left(\frac{du}{dt}\right)^2 + y = 1$.We can always check if a function is really a solution to a differetial equation

Slope Fields. In this section we learn a qualitative technique.

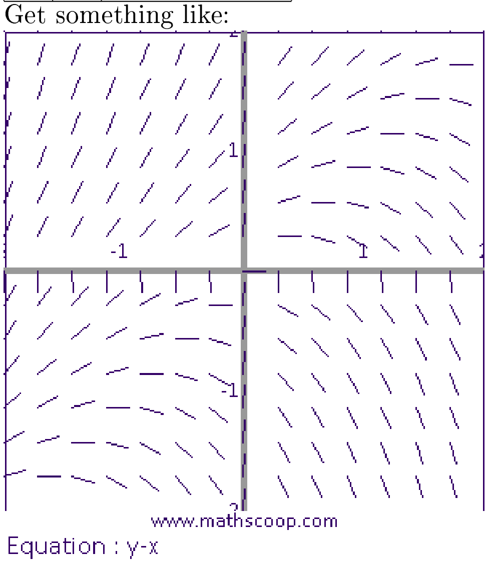
When we have an equation of the form $\frac{dy}{dt} = f(t, y)$. We can always make a **direction field/slope field**. A slope field contains minitangets at several points of a graph.

Example1: We want to make a 9 point slope field for $\frac{dy}{dt} = y - t$.

Make a table:

t	y	$f(t,y) = y - t$
-1	1	2
-1	0	1
-1	0	0
0	1	1
0	0	0
0	-1	-1
1	1	0
1	0	-1
1	-1	-2

Get something like:



- **Go over dfield with the class: This is an applet you can find on the main course webpage.**

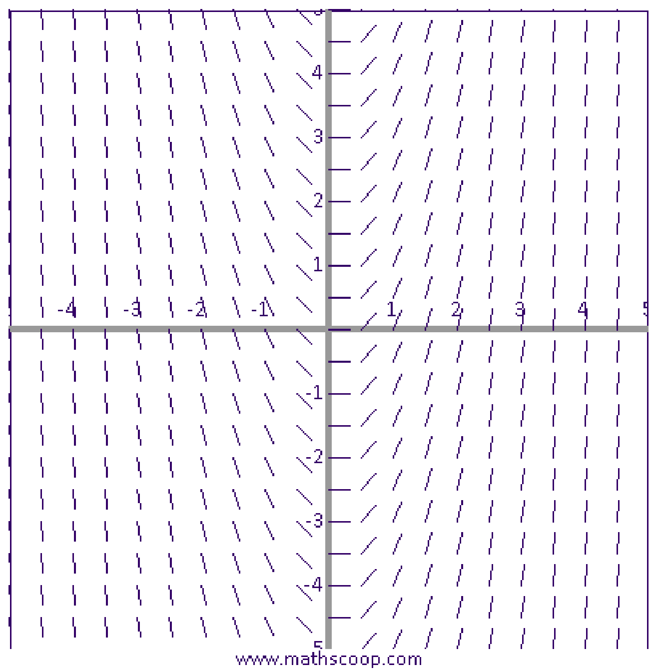
Two important cases:

They are of the form

$$\frac{dy}{dt} = f(t) \text{ or } \frac{dy}{dt} = f(y).$$

Type1: $\frac{dy}{dt} = f(t)$

- The slopes are always the same in each **vertical line**. Draw picture!



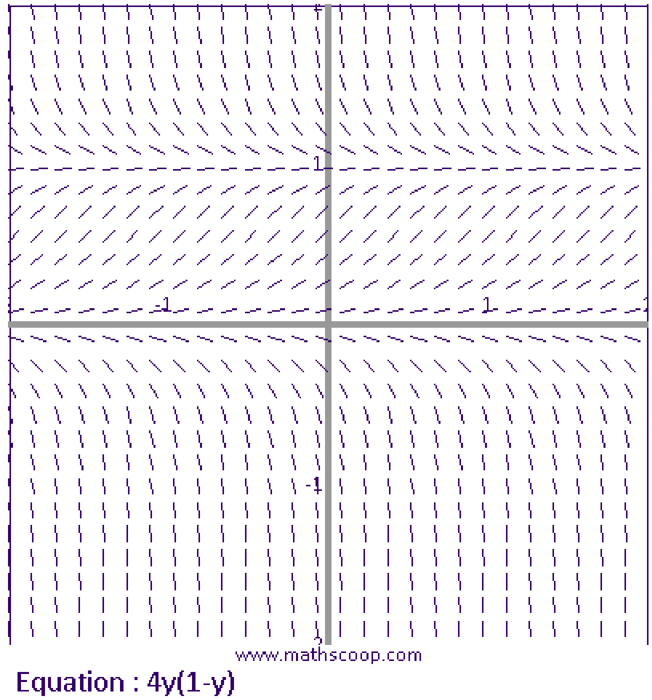
Equation : 2x

- Draw slope field of $\frac{dy}{dt} = 2t$ and get
- Find explicit solution and show it makes sense.

<u>**Type2:**</u> $\frac{dy}{dt} = f(y)$

- Recall that these were called autonomous equations.
- The slopes are always the same in each **horizontal** line. Draw picture!
- Draw a slope field of $\frac{dy}{dt} = 4y(1-y)$.
  - Always begin with the equilibrium solutions, $y = 0, 1$.
  - Then check slopes between there equilibrium solutions. Like:

$$\begin{aligned} y'(2) &= (-) \\ y'(.5) &= (+) \\ y'(-1) &= (-). \end{aligned}$$



and sketch something like this:

– Try to sketch a curve!

Matching slope fields:

When trying to match slope fields you should always follow these steps:

(1) Factor!
(2) Find the equilibrium solutions
(3) Test points between equilibrium solutions.

Do Worksheet with them!

Draw behavior given different initial conditions on the worksheet.

Look at Long term behavior! Ask questions on worksheet.

Sec. 2.1 - Integrating Factors

Integrating Factors Method:

Lets start with the linear diff eq:

$$\frac{dy}{dt} = a(t)y + b(t)$$

and rewrite it as

$$\frac{dy}{dt} + p(t)y = g(t)$$

where I let $g(t) = b(t)$ and $p(t) = -a(t)$. Then notice that $\frac{dy}{dt} + p(t)y$ looks awefully like a **product rule** of some sort. In the product rule, there are two functions. Clearly one function will be $y(t)$, but what will the second function be. We call $\mu(t)$ the **integrating factor** that makes the LHS into a product rule. Let's multiply both sides by $\mu(t)$ and get

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t),$$

then if we want the LHS to be a product rule then

$$LHS = \frac{d\left[\mu(t)y(t)\right]}{dt} = \mu(t)\frac{dy}{dt} + \mu(t)p(t)y.$$

Let's just assume this works for now and then find out what the $\mu(t)$ needs to be later. Setting the LHS to RHS we get

$$\frac{d\left[\mu(t)y(t)\right]}{dt} = \mu(t)g(t).$$

Then integrating we get

$$\int \frac{d\left[\mu(t)y(t)\right]}{dt}dt = \int \mu(t)g(t)dt.$$

But we know integrating cancels differentiation thus the LHS equals $\mu(t)y(t)$ so that

$$\mu(t)y(t) = \int \mu(t)g(t)dt. + C$$

and dividing by $\mu(t)$ we get that

$$\boxed{y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt + C\right].}$$

This is our **general solution.**

Find the integrating factor:

So recall that for the product rule to work we have

$$\frac{d\left[\mu(t)y(t)\right]}{dt} = \mu(t)\frac{dy}{dt} + \mu(t)p(t)y$$

but then this only happens if the derivative of $\mu(t)$ is $\mu(t)g(t)$ (by product rule!!!!) thus

$$\frac{d\left[\mu(t)\right]}{dt} = \mu(t)p(t).$$

Rewrite this as

$$\frac{d\mu}{dt} = \mu p$$

which is a separable equation and thus

$$\int \frac{d\mu}{\mu} = \int p(t)dt \iff \ln|\mu| = \int p(t)dt$$
$$\iff \mu = e^{\int p(t)dt}.$$

Thus $\mu = e^{\int p(t)dt}$ and we don't need a constant here because we only need ONE integrating factor.

Summarize:

So basically if we know how to integrate $\int \mu(t)g(t)dt$ then the general solution of

$$\frac{dy}{dt} + p(t)y = g(t)$$

will be

$$\boxed{y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt. + C\right] \text{ where } \mu(t) = e^{\int p(t)dt}.}$$

Example1(without formula): Find general solution of $\frac{dy}{dt} = \frac{3}{t}y + t^5$.

- **Step 1:** Rewrite as

$$\frac{dy}{dt} - \frac{3}{t}y = t^5$$

so that $p(t) = -\frac{3}{t}$ and $g(t) = t^5$.

- **Step 2:** Find an integrating factor:

$$\mu(t) = e^{\int -\frac{3}{t}dt} = e^{-3\ln t} = t^{-3} = \frac{1}{t^3}.$$

Note we only need an integrating factor, not a general integratin factor. So we never need to have a $+C$ in this step !!!!!! In the next step we will that we also don't need the asbolute value inside the ln.

- **Step3:** Multiply BOTH SIDES of the equation by $\mu(t)$ and get

$$\frac{1}{t^3}\frac{dy}{dt} - \frac{3}{t^4}y = t^2$$

and notice that

$$\frac{1}{t^3}\frac{dy}{dt} - \frac{3}{t^4}y = t^2$$
$$||$$
$$\frac{d\left[\frac{1}{t^3}y\right]}{dt} = t^2$$

- **Step4:** Integrate and solve for $y(t)$ (don't forget the constant C in this step, very important)

$$\int \frac{d\left[\frac{1}{t^3}y\right]}{dt} = \int t^2 dt + C \iff \frac{1}{t^3}y = \frac{t^3}{3} + C$$
$$\iff y(t) = \frac{t^6}{3} + Ct^3.$$

Example1(with formula): Solve the IVP: $\frac{dy}{dt} = \frac{3}{t}y + t^5$ with $y(1) = \frac{4}{3}$. In this eaxmple we'll skip the previous steps and go straight to using the formula.

- **Step 1:** Rewrite as
$$\frac{dy}{dt} - \frac{3}{t}y = t^5$$
so that $p(t) = -\frac{3}{t}$ and $g(t) = t^5$.
- **Step 2:** Find an integrating factor:
$$\mu(t) = e^{\int -\frac{3}{t}dt} = e^{-3\ln t} = t^{-3} = \frac{1}{t^3}.$$
- **Step3:** I can just plug in the formula and get
$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt. + C\right] \\ &= t^3\left[\int \frac{1}{t^3}t^5 dt + C\right] \\ &= t^3\left[\frac{t^3}{3} + C\right] = \frac{1}{3}t^6 + Ct^3. \end{aligned}$$
- **Step4:** Since $y(2) = \frac{4}{3}$ then
$$\frac{4}{3} = \frac{1}{3} + C$$
so $C = 1$ so that
$$y(t) = \frac{1}{3}t^6 + t^3.$$

Example2 (using formula): Find general solution $\frac{dy}{dt} = y + 9\cos t^2$.

- **Step 1:** Rewrite as
$$\frac{dy}{dt} - y = 9\cos t^2$$
so that $p(t) = -1$ and $g(t) = 9\cos t^2$.
- **Step 2:** Find an integrating factor:
$$\mu(t) = e^{\int -1 dt} = e^{-t}.$$
Note we only need an integrating factor, not a general integratin factor. So we never need to have a $+C$ in this step !!!!!!
- **Step3:** I can go through the process again, or I can just plug in the formula and get
$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt. + C\right] \\ &= \frac{1}{e^{-t}}\left[\int e^{-t}9\cos t^2 dt + C\right] \\ &= e^t\left[\int e^{-t}9\cos t^2 dt + C\right] \end{aligned}$$
Can't integrate, so we write the answer this way.

Example3 (it time permits): Find general solution of $t^3y' + 4t^2y = e^{-t}$

- **Step 1:** Rewrite as
$$y' + \frac{4}{t}y = \frac{e^{-t}}{t^3}$$
so that $p(t) = 4t$ and $g(t) = \frac{e^{-t}}{t^3}$.

- **Step 2:** Find an integrating factor:
$$\mu(t) = e^{\int \frac{4}{t}dt} = e^{4\ln|t|} = t^4.$$
Note we only need an integrating factor, not a general integratin factor. So we never need to have a $+C$ in this step !!!!!!
- **Step3:** I can go through the process again, or I can just plug in the formula and get
$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)g(t)dt. + C\right] \\ &= \frac{1}{t^4}\left[\int t^4 \frac{e^{-t}}{t^3} dt + C\right] \\ &= \frac{1}{t^4}\left[\int te^{-t}dt + C\right] \\ &= \frac{1}{t^4}\left[-te^{-t} - e^{-t} + C\right] \\ &= -\frac{1}{t^3}e^{-t} - \frac{1}{t^4}e^{-t} + \frac{C}{t^4}. \end{aligned}$$

where I used integration by parts.

Sec. 2.2 Separable Equations

One of the easiest method to solve a first order ODE is to separate the variable. We've done quite a bit already in the previous sections. We can only do this when the ODE is of the form

$$\frac{dy}{dt} = g(t)h(y). \qquad (1)$$

Separate the variables and get

$$\frac{dy}{h(y)} = g(t)dt,$$

and then integrate both side with respect to their respective variable. This is legal by a u-substitution argument. (This is informal algebra!)

Sometime you'll see it written of the following form

$$M(x)dy + N(y)dy = 0.$$

and that is your solution. If you can solve for y then do it.

- A separable equation is said to be **autonomous** if the RHS of is dependent only on ym meaning

$$\frac{dy}{dt} = f(y).$$

  - Like $\frac{dy}{dt} = \frac{y}{y^2+1}$
  - But NOT $\frac{dy}{dt} = ty$.

Example1: Notice that $\frac{dy}{dt} = y + t$ is not separable

- But we can solve this using the methods of the previous section.

Example2: $\frac{dy}{dt} = \frac{t}{y^2}$

Then do:

$$\begin{aligned} \frac{dy}{dt} = \frac{t}{y^2} \quad &\Longleftrightarrow \quad y^2 dy = tdt \\ &\Longleftrightarrow \quad \int y^2 dy = \int tdt \\ &\Longleftrightarrow \quad \frac{y^3}{3} = \frac{t^2}{2} + C \\ &\Longleftrightarrow \quad y = \sqrt[3]{\frac{3t^2}{2} + 3c}. \end{aligned}$$

This is an explicit solution:

Example 3: Find the general soltuon for $\frac{dy}{dt} = y^2$ (Missing solution)

- First let's find the equilibrium solutions: $y(t) = 0$ is the only one.
- Then do use the general procedure

$$\begin{aligned} \frac{dy}{dt} = y^2 \quad &\Longleftrightarrow \quad \frac{1}{y^2}dy = dt \\ &\Longleftrightarrow \quad \int \frac{1}{y^2}dy = \int dt \\ &\Longleftrightarrow \quad -\frac{1}{y} = t + C. \end{aligned}$$

But notice that

$$y = -\frac{1}{t+C}$$

does NOT solve the IVP with $y(0) = 0$. Thus we have to include the equilibrium solution $y(t) = 0$, to get the complete Geberal Solution.

Moral of the story: Always find the equilibrium solutions first!!!!!!!!

Example 5: Solve the IVP (Clever quadratic formula trick)

$$\frac{dy}{dx} = \frac{2x+1}{y+1} \quad y(0) = 1$$

- Note there are no equilibrium solutions
- Then do use the general procedure

$$\begin{aligned}
\frac{dy}{dt} = \frac{2x+1}{y+1} \quad &\iff \quad \int (y+1)\, dy = \int (2x+1)\, dx \\
&\iff \quad \frac{y^2}{2} + y = x^2 + x + C \\
&\iff \quad \frac{y^2}{2} + y - x^2 - x + c = 0 \\
&\iff \quad y^2 + 2y - 2x^2 - 2x + C = 0
\end{aligned}$$

Then we can use the quadratic formula on

$$ay^2 + by + c = 0$$

where

$$\begin{aligned}
a &= 1 \\
b &= 2 \\
c &= -2x^2 - 2x + C
\end{aligned}$$

hence

$$\begin{aligned}
y &= \frac{-2 \pm \sqrt{4 - 4\left(-2x^2 - 2x + C\right)}}{2} \\
&= -1 \pm \sqrt{1 + 2x^2 + 2x + C} \\
&= -1 \pm \sqrt{2x^2 + 2x + C}
\end{aligned}$$

- Now to solve the IVP $y(0) = 1$

$$1 = y(0) = -1 \pm \sqrt{C} =$$

so that

$$2 = \pm\sqrt{c}$$

since the LHS is Positive we choose the positive sign in the $\pm$ so that

$$2 = \sqrt{c}$$

hence

$$c = 4$$

so that

$$y(x) = -1 + \sqrt{2x^2 + 2x + 4}.$$

Example 6: Find the general solution for $\frac{dy}{dt} = \frac{y}{1+y^2}$ (Implicit Solutions (we get stuck))
Get

$$\ln|y| + \frac{y^2}{2} = t + C$$

and leave it that way as there is no nice way to solve this. But any function $y(t)$ that satisfies the equation about is a solution to our ODE.

Some more examples:

- $\frac{dy}{dt} = t^4 y$ with $y(0) = 1$
  - Start with equilibrium solutions $y = 0$.
  - Get $|y| = Ce^{t^5/5}$ but notice that by choise of C this shortens to $y = Ce^{t^5/5}$.
  - Get $y = e^{t^5/5}$.
- $\frac{dy}{dt} = (y+1)(y+5)$
  - Start with equilibrium solutions $y = -1, -5$.
  - Use Partial fractions to get $\frac{1}{(y+1)(y+5)} = \frac{1/4}{y+1} - \frac{1/4}{y+5}$
  - Solution is

$$\begin{aligned}\frac{1}{4}\ln|y+1| - \frac{1}{4}\ln|y+5| = t + C \quad &\iff \quad \ln\left|\frac{y+1}{y+5}\right| = 4t + C_1\\ &\iff \quad \left|\frac{y+1}{y+5}\right| = C_2 e^{4t}\\ &\iff \quad \frac{y+1}{y+5} = C_3 e^{4t}\\ &\iff \quad y = \frac{5ke^{4t} - 1}{1 - ke^{4t}}.\end{aligned}$$

  - This yields all solutions but the equilibrium solutions.

Sec. 2.2(b) Separable Homogeneous Equations using substitution

- Consider an ODE
$$\frac{dy}{dx}=f(x,y)$$
and suppose we can rewrite it in the form
$$\frac{dy}{dx}=F\left(\frac{y}{x}\right).$$
  - Then we can define a new variable
$$v=\frac{y}{x}\quad\text{(Important)}$$
and write everything in terms of only v and x.
  - Note that by implicit differentiation
$$\frac{dv}{dx}=\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y$$
but since
$$y=vx$$
then
$$\frac{dv}{dx}=\frac{1}{x}\frac{dy}{dx}-\frac{v}{x}$$
and multiplying everything by x and solving for $\frac{dy}{dx}$ we have
$$\frac{dy}{dx}=x\frac{dv}{dx}+v\quad\text{(Important)}$$
- **Example1:** Consider
$$\frac{dy}{dx}=\frac{x^2+xy+y^2}{x^2}.$$
  - **Part(a):** Show that this ODE is homogeneous and rewrite the entire equation by only v and x.
    * To see this we divide the numerator and denominator by x^2 and get
$$\frac{dy}{dx}=\frac{1+\frac{y}{x}+\left(\frac{y}{x}\right)^2}{1}$$
and replacing $\frac{dy}{dx}=x\frac{dv}{dx}+v$ and $v=\frac{y}{x}$ we get a new equation
$$x\frac{dv}{dx}+v\frac{dy}{dx}=1+v+v^2.$$
  - **Part(b):** Solve the ODE in terms of v and then return everything into terms of y,x.
    * We rewrite
$$\begin{aligned}x\frac{dv}{dx}=1+v^2 &\iff \int\frac{dv}{1+v^2}=\int\frac{1}{x}dx\\ &\iff \tan^{-1}(v)=\ln|x|+c\\ &\iff \tan^{-1}\left(\frac{y}{x}\right)=\ln|x|+c.\end{aligned}$$
- **Example2: (previous exam problem)** Find the solution to
$$y'=\frac{y}{x}+\frac{x}{y}\quad ,x>0.$$

- **Step1:** First since if you can apply any of the method of the previous sections (linear? separable?) But notice that this is homogeneous for if I let $v = \frac{y}{x}$ then
$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{y/x} = v + \frac{1}{v}.$$
- **Step2:** Recall that $\frac{dy}{dx} = x\frac{dv}{dx} + v$ so that
$$\begin{aligned} x\frac{dv}{dx} + v = v + \frac{1}{v} &\iff \int v dv = \int \frac{1}{x} dx \\ &\iff \frac{v^2}{2} = \ln|x| + C \\ &\iff y^2 = 2x^2 \ln|x| + kx^2. \end{aligned}$$

• **Example3: (not always the same substitution)** Rewrite the equation
$$\frac{dy}{dx} = e^{9y-x}$$
in terms of only v, x by letting $v = 9y - x$.
- **Solution:** Using implicit differentiation on $v = 9y - x$ we have
$$\frac{dv}{dx} = 9\frac{dy}{dx} - 1$$
so that
$$\frac{dy}{dx} = \frac{1}{9}\frac{dv}{dx} + \frac{1}{9}$$
hence
$$\begin{aligned} \frac{dy}{dx} = e^{9y-x} &\iff \frac{1}{9}\frac{dv}{dx} + \frac{1}{9} = e^v \\ &\iff \frac{dy}{dx} = 9e^v - 1. \end{aligned}$$
and this can be easily solved by separating variables.

Section 2.3 Modeling with diffential equations

- **What are Differential Equations used for?**
- This is class is about predicting the future. Meteorologists try to do it all the time with the weather, and they get it wrong all the time. This means modeling is hard.
- Three aproaches
  - Analytic: explicit solutions
  - Qualitative: Use geometry to see long term behavious. For example, the population is increasing.
  - Numerical: Approximations to actual solutions.
- Model Building:
  - 1. State Assumptions (science step, Newton's law of motions, etc, ...)
    - 2. Describe variables, paramters: Indepedent variable(t), depedent variable, parameters(do not change with time)
    - 3. Create Equations: Rate of change, slope: Derivative.
      - the word "is" means equal.
      - A is proportional to B means $A = kB$.

  Example: Population growth
  - **Goal:** Want to write a differetial equation that models population growth of say Zebras.
  - **Assumption:** The rate of growth of the population is proportional to the size of the population.
  - **Problem:** Write a differential equation that governs this
    - Let $P(t)$ be the population of zebras at time t.

    So for now we have

$$\frac{dP}{dt} = k \cdot P.$$

    - Note here that k is a **parameter** that can be changes once we know more information.
    - For example if we know the proportion is $k = 2$, then

$$\frac{dP}{dt} = k \cdot P$$

      and we already saw earlier that $P(t) = 9e^{2t}$ is a solution to this.

Mixing Problem1:

Problem. A vat contains 60**L** of water with 5 **kg** of salt water dissolved in it. A salt water solution that contains 2 **kg** of salt per liter enters the vat at a rate of 3 **L/min**. Pure water is also flowing into the vat at a rate of 2 **L/min.** The solution in the vat is kept well mixed and is drained at a rate of 5 **L/min,** so that the rate in is the same as the rate out. Thus there is always 60**L** of salt water at any given time. How much still remains after 30 minutes? What is the long term behavior?

Solution:
Step1: Define variables
Let $y(t)$ =amount of salt at time t. Let $y(0) = 5$ kg.
Step2: Find Rate in/ Rate out
Note that for anything that comes in you can always find the Rate In as

$$\text{Rate in} = \begin{pmatrix} \text{concentrarion} \\ \text{of stuff coming in} \end{pmatrix} \times \text{Rate}.$$

Similarly you can always find the Rate out as

$$\text{Rate out} = \begin{pmatrix} \text{concentrarion} \\ \text{of stuff going out} \end{pmatrix} \times \text{Rate}.$$

Using the information from the problem we have

$$\begin{aligned} \text{Rate in} \quad &= \quad \underbrace{\left(2\frac{\text{kg}}{\text{L}}\right)\left(3\frac{\text{L}}{\text{min}}\right)}_{\text{-salt water solution}} + \underbrace{\left(0\frac{\text{kg}}{L}\right)\left(2\frac{\text{L}}{\text{min}}\right)}_{\text{–pure water}} \\ &= \quad 6\frac{\text{kg}}{\text{min}}. \end{aligned}$$

and

$$\begin{aligned} \text{Rate out} \quad &= \quad \begin{pmatrix} \text{concentrarion} \\ \text{of stuff going out} \end{pmatrix} \times \text{Rate} \\ &= \quad \left(\frac{y(t)}{60}\frac{kg}{L}\right) \times 5\frac{\text{L}}{\text{min}}. \\ &= \quad \frac{y(t)}{12}\frac{kg}{min}. \end{aligned}$$

Step 3: Write the IVP
Always recall that for mixing problems we have

$$\begin{aligned} \frac{dy}{dt} \quad &= \quad \text{Rate in} - \text{Rate out} \\ &= \quad 6 - \frac{y}{12}. \end{aligned}$$

and the initial condition i

$$y(0) = 5.$$

Step 4: Find the common denominator and solve using separation of variables.
Write

$$\frac{dy}{dt} = 6 - \frac{y}{12} = \frac{72 - y}{12}$$

and using separation of variables we get

$$\begin{aligned} \frac{dy}{dt} = \frac{72-y}{12} \quad &\Longleftrightarrow \quad \frac{dy}{72-y} = \frac{dt}{12} \\ &\Longleftrightarrow \quad -\ln|72-y| = \frac{t}{12} + C_1 \\ &\Longleftrightarrow \quad \ln|72-y| = \frac{-t}{12} + C_2 \\ &\Longleftrightarrow \quad |72-y| = C_3 e^{-\frac{t}{12}} \\ &\Longleftrightarrow \quad 72 - y = ke^{-\frac{t}{12}} \\ &\Longleftrightarrow \quad y = 72 - ke^{-\frac{t}{12}}. \end{aligned}$$

Solving the IVP by using $y(0) = 5$ to get When we have an equation of the form $\frac{dy}{dt} = f(t, y)$. We can always make a slope field. A slope field contains minitangets at several points of a graph.

Mixing Problem #2:

- The difference here is that now we allow the total volume of fluid to vary, when before it was kept fixed.

Problem. A 400-**gallon** tank initially contains 200 **gallons** of water containing 3 **pounds of sugar per gallon**. Suppose water containing 5 **pounds per gallon** flows into the the top of the tank at a rate of 6 **gallons per minute.** The water in the tank is kept well mixed, and 4 **gallons per minute** are removed from the bottom of the tank. How much sugar is in the tank when the tank is full?

Solution:
Step1: Define variables
Let $y(t)$ = amount of sugar at time t, which is in minutes. Let $y(0) = 3 \times 200 = 600$ pounds.
Step2: Find Rate in/ Rate out
Note that for anything that comes in you can always find the Rate In as

$$\text{Rate in} = \begin{pmatrix} \text{concentration} \\ \text{of sugar coming in} \end{pmatrix} \times \text{Rate}.$$

Similarly you can always find the Rate out as

$$\text{Rate out} = \begin{pmatrix} \text{concentrartion} \\ \text{of sugar coming out} \end{pmatrix} \times \text{Rate}.$$

We have

$$\begin{aligned} \textbf{Rate in} &= \underbrace{\left(5\frac{\text{pounds}}{\text{gallon}}\right)\left(6\frac{\text{gallons}}{\text{min}}\right)}_{\text{-sugar water solution}} \\ &= 30\frac{\text{pounds}}{\text{gallon}}. \end{aligned}$$

To find the concentration of sugar coming out we have know that the amount of water at time t.

$$\begin{aligned} \text{Water at timt } t &= 200 \text{ gallons} + \left(6\frac{\text{gallons}}{\text{min}} - 4\frac{\text{gallons}}{\text{min}}\right) t \\ &= 200 + 2t, \end{aligned}$$

So

$$\begin{aligned} \textbf{Rate out} &= \begin{pmatrix} \text{concentrarion} \\ \text{of stuff going out} \end{pmatrix} \times \text{Rate} \\ &= \left(\frac{y(t)}{200+2t}\frac{pounds}{gallon}\right) \times 4\frac{\text{gallons}}{\text{min}}. \\ &= 4\frac{y(t)}{200+2t}\frac{pound}{min}. \end{aligned}$$

Step 3: Write the IVP
Always recall that for mixing problems we have

$$\begin{aligned} \frac{dy}{dt} &= \text{Rate in} - \text{Rate out} \\ &= 30 - \frac{4}{200+2t}y. \end{aligned}$$

and the initial condition

$$y(0) = 600.$$

Step 4: Solve using the Method of integrating factors:
Write

$$\frac{dy}{dt} + \frac{4}{200+2t}y = 30$$

so that $g(t) = \frac{4}{200+2t}$ and $b(t) = 30$. Thus the integrating factor is

$$\mu(t) = e^{4\int \frac{dt}{200+2t}} = e^{2\int \frac{dt}{100+t}} = e^{2\ln(100+t)} = (100+t)^2 .$$

Thus using the formula I have that

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)b(t)dt. + C\right] \\ &= \frac{1}{(100+t)^2}\left[30\int (100+t)^2 \, dt. + C\right] \\ &= \frac{1}{(100+t)^2}\left[30\frac{(100+t)^3}{3}. + C\right] \\ &= \frac{1}{(100+t)^2}\left[10\,(100+t)^3 + C\right] \end{aligned}$$

using $y(0) = 600$ we get that

$$600 = \frac{1}{100^2}\left[10 \cdot 100^3 + C\right]$$

so that

$$C = -4,000,000$$

thus

$$y(t) = \frac{10\,(100+t)^3 - 4,000,000}{(100+t)^2}.$$

Step5: Answer the question
Since the amount of water in the tank is $200 + 2t$ then it fills up when

$$200 + 2t = 400$$

so that $t = 100$. Thus the amount of sugar is

$$\begin{aligned} y(100) &= \frac{10\,(200)^3 - 4,000,000}{(200)^2} \\ &= 1,900 \text{ pounds.} \end{aligned}$$

Section 2.3 Modeling with diffential equations - More Problems

- **Physics Problem:**
- We will consider problems involving either
  - **free-fall** or
  - **throwing up** an object straight up in the air
- We will also consider when there is some "**air resistance of magnitude** $R(v)$ **directed opposite to the velocity** v".
- Setting our equation
  - Since we know that $F = \text{mass} \times \text{accelaration} = m\frac{dv}{dt}$. This will always be our LHS=left hand side of our equation.
  - The RHS depends on the problem given (e.g. free fall, throwing object up? is there resistance?)
  - Thus our Equations will be in the form

$$m\frac{dv}{dt} = \pm R(v) \pm mg$$

    * We'll have $-R(v)$: If object is going up, i.e. $v > 0$. (Since air resistance $R(v)$ is directed opposite to the velocity v)
    * We'll have $+R(v)$: If object is going down, i.e. $v < 0$ (Since air resistance $R(v)$ is directed opposite to the velocity v)
    * We'll have $-mg$: If the object was thown up. (Which means the force is going opposite the natural free fall state)
    * We'll have mg: If the object was released in free fall.
- **Example1:** Suppose a rocket with mass 10 kg is launched upward with initial velocity 20 m/s from a platform that is 3 meters high. Suppose there is a force due to air resistance of magnitude $|v|$ directed opposite to the velocity, where the velocity v is measured in m/s. We neglect the variation of the earth's gravitational fields with distance. (Since it's not going very high anyways)
  - **Part (a):** Find the maximum height above the ground that the rocket reaches.
    * **Solution:** Suppose we consider when the rocket is going up in the air before it has reached the maximum height. Let $R(v) = |v|$ be the resistance, then using what've discussed above we have

$$m\frac{dv}{dt} = -R(v) - mg,$$

and we have $-R(v)$ since the rocket is still going up, and $-mg$ since the rocket was launched upwards (the negative because the force is going againsts its natural gravitational pull). Thus since the rocket is going up then $v > 0$. Recall that

$$|v| = \begin{cases} v & v > 0 \\ -v & v < 0 \end{cases}$$

then

$$m\frac{dv}{dt} = -|v| - mg = -v - mg.$$

Hence

$$m\frac{dv}{dt} = -v - mg.$$

* Solving this we have that

$$\int \frac{dv}{v+mg} = \frac{\int -dt}{m} \iff \ln|v+mg| = -\frac{t}{m} + C$$
$$\iff |v+mg| = Ce^{-t/m}$$
$$\iff v+mg = Ce^{-t/m}$$
$$\iff v = Ce^{-t/m} - mg.$$

* Since $v(0) = 20$ Then we can solve for C and obtain (using $g = 9.8$ m/s^2)

$$v(t) = (20+mg)\,e^{-t/m} - mg,$$
$$= 118e^{-t/10} - 98.$$

and this equation is valid only when the rocket is going up.

* The maximum happens when velocity is equal to zero. Thus set $v(t_1) = 0$ and we get that

$$0 = 118e^{-t/10} - 98 \iff t_1 = -10\ln\left(\frac{98}{118}\right)$$
$$\iff t_1 \approx 1.86.$$

* Solve for position: We get

$$x(t) = \int v(t)dt + C$$
$$= -1180e^{-t/10} - 98t + C.$$

Since $x(0) = 3$, then

$$3 = -1180e^0 - 98\cdot 0 + C \iff 3 = -1180 + C$$
$$\iff C = 1183.$$

* Thus

$$x(t) = -1180e^{-t/10} - 98t + 1183.$$

Then

$$\text{maximum height} = x(1.86)$$
$$\approx 21.$$

– **Part (b):** Find the time that the rocket hits the ground. Assuming it missed the platform.
  * **Solution:** We need to find the equation of when the rocket is falling down.When the rocket is falling down we thus have the following equation:

$$m\frac{dv}{dt} = R(v) - mg,$$

and we have $R(v)$ since the rocket is going down, and $-mg$ since the rocket was launched upwards (againsts its natural gravitational pull). Thus since the rocket is going **down** then $v < 0$. Recall that

$$|v| = \begin{cases} v & v > 0 \\ -v & v < 0 \end{cases}$$

then $|v| = -v$ so that

$$m\frac{dv}{dt} = |v| - mg = -v - mg.$$

hence

$$m\frac{dv}{dt} = -v - mg.$$

- Solving this we have that $v_2(t) = Ce^{-t/m} - mg$ with initial condition $v_2(0) = 0$. Thus

$$\begin{aligned} v_2(t) &= mge^{-t/m} - mg \\ &= 98e^{-t/10} - 98. \end{aligned}$$

- Then

$$\begin{aligned} x_2(t) &= \int v_2(t)dt + C \\ &= -980e^{-t/10} - 98t + C \end{aligned}$$

since

$$x_2(0) = \text{maximum height} = 21$$

then solving for C we have

$$x_2(t) = -980e^{-t/10} - 98t + 1001.$$

- To find out when $x_2(t)$ hits the ground we need to find t_2 such that $x_2(t_2) = 0$ thus (using a calculator)

$$0 = -980e^{-t_2/10} - 98t_2 + 1001 \iff t_2 \approx 2.14.$$

- Thus the ball hits the ground by adding the time it takes to reach its maximum plus the time after that:

$$t_0 = t_1 + t_2 = 1.86 + 2.14 = 4 \text{ seconds}.$$

- **Example2:** Consider the same scenario as before. A rocket with mass 10 kg is launched upward with initial velocity 20 m/s from a platform that is 3 meters high. Except, there is a force due to air resistance of magnitude $v^2/5$ directed opposite to the velocity, where the velocity v is measured in m/s.
  - **Part(a):** Write the differential equation for velocity, when the rocket is still going up:
    - **Solution:** Let $R(v) = v^2/5$ be the resistance, then

$$m\frac{dv}{dt} = -R(v) - mg,$$

and we have $-R(v)$ since the rocket is still going up, and $-mg$ since the rocket was launched upwards (againsts its natural graviational pull). Thus

$$m\frac{dv}{dt} = -\frac{v^2}{5} - mg \iff m\frac{dv}{dt} = -\frac{v^2}{5} - 98$$

  - **Part(b):** Write the differential equation for velocity, when the rocket has already reched maximum and is already going down.

   * **Solution:** Let $R(v) = v^2/5$ be the resistance, then using the above we have

$$m\frac{dv}{dt} = R(v) - mg,$$

   and we have $R(v)$ since the rocket is going down, and $-mg$ since the rocket was launched upwards (againsts its natural graviational pull). Thus

$$m\frac{dv}{dt} = \frac{v^2}{5} - 98.$$

- **Example3:** Suppose we fly a plane at an altitude of 5000 ft and drop a watermelon that weighs 64 pounds vertically downward. Assume that the force of air resistance, which is directed opposite to the velocity, is of magnitude $|v|/128$. (Use $g = 32$ ft/sec^2)
  - **Question:** Find how long it takes for the watermelon to hit the ground?
  - **Solution:** Here we assume the positive direction is down. Thus $v > 0$ as the object falls, hence

$$m\frac{dv}{dt} = -R(v) + mg,$$

  and we have $-R(v)$ since the watermellon is going down (which is in the positive direction), and $+mg$ since the rocket is being dropped by freefall. Now recall that

$$\text{weight} = mg$$

  then $m = \frac{64}{32} = 2$.
  - Then since the watermelon is going down then $v > 0$, so that $R(v) = 128\,|v| = 128v$,

$$\begin{aligned} m\frac{dv}{dt} = \frac{-v}{128} + mg &\iff 2\frac{dv}{dt} = -\frac{v}{128} + 64 \\ &\iff \frac{dv}{dt} = -\frac{v}{256} + 32 \\ &\iff \int \frac{dv}{v - 256 \cdot 32} = -\int \frac{1}{256}dt \\ &\iff v(t) = Ce^{-t/256} + 256 \cdot 32. \end{aligned}$$

  and since $v(0) = 0$ then

$$v(t) = -256 \cdot 32e^{-t/256} + 256 \cdot 32.$$

  - Solving for the distance traveled $x(t)$ from the ground we have

$$x(t) = (256)^2 \cdot 32e^{-t/256} + 32 \cdot (256)\, t + C$$

  and letting $x(0) = 0$, then

$$x(t) = (256)^2 \cdot 32e^{-t/256} + 32 \cdot (256)\, t - (256)^2 \cdot 32$$

  Then

$$x(t_0) = 5000 \iff t_0 \approx 17.88 \text{ seconds}.$$

- **Example4:** Newton's Law of Cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and its surroundings. Suppose there was a murder in a room that is 70° F. Assume the victim had a temperature of 98.6° when murded and 2 hours after the murder the body had a temperature of 50°F. Find how long it will take for the body to reach 20°F.

- **Solution:** One needs to solve the following IVP: Let $T(t)$ be the temperature of the victim, then
$$\frac{dT}{dt}=k\left(T-70\right),\quad T\left(0\right)=98.6$$
and need to use the information $T(2)=50$ to solve for k.

Section 2.4 Exitence and Uniqueness of Solutions

We want to know if solutions even exists to an ODE.

- If this models a physical phenomona and no solutions exists, then there is something seriously wrong about your model.
  - Why spend time trying to find a solution, and doing all the things in previous sections if no solutions exist.
- **Example:** Suppose we have
$$2x^5 - 10x + 3 = 0.$$
Plugging $x = \pm 1$ into $f(x) = 2x^5 - 10x + 5$ we get $f(1) = -5$ and $f(-1) = 11$.
  - We draw a continuous sketch of this graph, and show it must cross the $x-$axis.
  - By the intermediate value theorem we know that at least one solution exists.Since somewhere in between it must have $x = -1$ and $x = 1$ the function $f(x)$ must have crossed the $x-$ axis.
  - There could be more than one, we'd like to know if we should stop searching for more solutions.
  - No "quadratic formula" for 5th degree polynomials.
- **Example:** No solutions for $x^2 + 1 = 0$.

Theorem. *1 (Linear* 1*st order ODE) If the function p and g are continuous on an open interval* $I = (a, b)$ *containing the point* $t = t_0$*, then there exists a unique function* $y = \phi(t)$ *that satisties the ODE*
$$y' + p(t)y = g(t)$$
for each t in I, and that also satisfies the intial condition
$$y(t_0) = y_0$$
where y_0 *is an arbitrary initial value.*

- This theorem guarantees the existence and uniqueness of solutions under the assumption of the theorem.
- This is only for IVP, nothing to do with separate solutions to ODE's. (which we already know there are many)
- **Important:** This theorem allows you to know the domain before even solving for the solution
- **Example1:**
  - **Part (a):** Without solving the problem, what is the largest interval in which the solution of the given IVP is certain to exist?
$$(t-1)y' + \cos t y = \frac{e^t}{t-6} \quad y(3) = -4$$
  - **Solution:** We rewrite as
$$y' + \frac{\cos t}{(t-1)} y = \frac{e^t}{(t-6)(t-1)}$$
Since $\frac{\cos t}{(t-1)}$ and $\frac{e^t}{(t-6)(t-1)}$ are only both continuous on $I = (1, 6)$ and since 3 is in I. Then we know there exists a unique solution $y = \phi(t)$ on the interval $(1, 6)$.
  - **Part(b):** What if I change the initial condition to
$$y(8) = 7,$$
then what is I?
    - **Solution:** Then $I = (6, \infty)$.

Theorem. *2 (General 1st Order ODE) Suppose $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous functions in a rectangle of the form*

$$\{(t,y) \mid a < t < b, c < y < d\} \quad (draw\ pic)$$

in the ty−plane.. If (t_0, y_0) is a point inside the rectangle. then there exists a unique $\epsilon > 0$ and a ***unique function*** *$y(t) = \phi(t)$ defined for $(t_0 - \epsilon, t_0 + \epsilon)$ that solves the initial value problem*

$$\frac{dy}{dt} = f(t,y), \qquad y(t_0) = y_0.$$

- **Warning:** Unlike Theorem 1, Theorem 2 does not tell you what domain the solution will be valid for. In this case, you really do have to solve for the solution to figure out the domain of the function.

Corollary. *Moreover assuming the same conditions as Theorem 1, if (t_0, y_0) is a point in this rectangle and if $y_1(t)$ and $y_2(t)$ are two functions that solve the IVP*

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0,$$

for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$, then

$$y_1(t) = y_2(t)$$

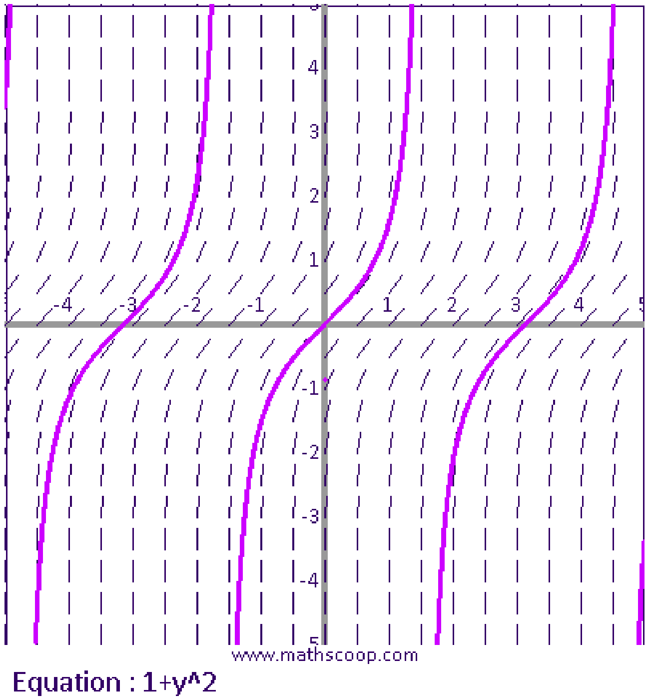
for $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

The Domain of solutions:

- Remember what a partial derivative means? For example, take the partials of $y^2 + t^2, yt$ and $y^2 t$.
- Notice that the Theorem only gives you a function $y(t)$ defined for some interval $(t_0 - \epsilon, t_0 + \epsilon)$.
  - the $\epsilon > 0$ may be super small,
  - so it may not be valid for big t. So effects how we can apply this real world solutions.
- **Example2:** Consider

$$\frac{dy}{dt} = 1 + y^2 \qquad y(0) = 0.$$

  - **Part (a):** Find where in the t-y plane the hypothesis of Theorem 2 is satisfied:
    * **Solution:** Note that $f(t,y) = 1 + y^2$ and $\frac{\partial f}{\partial y} = 2y$ are always continuous, thus satisfied in all of $\mathbb{R}^2$.
  - **Part (b):** Find the actual interval in which the IVP exists uniquely:
  - What if the solution blows up? Solve using separable equations and get $y(t) = \tan(t + c)$ but with initial condition you get $y(t) = \tan(t)$. But this solution is only valid for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



- Graph: :
- **Moral:** Unlike Theorem1, Theorem 2 does not says it needs to exits in the entire rectangle, it just says that there exists some interval in which it exists.

Example3: (Lack of uniqueness)

- Take $\frac{dy}{dt} = 3y^{2/3}$ and $y(0) = 0$.
  - **Question1:** Show that $y_1(t) = 0$ and $y_2(t) = t^3$ are two solutions to this IVP. Why does this not contradict Theorem 2?
    - **Solution:**We know the equilibrium $y_1(t) = 0$ which solves the IVP is one solution.. Use separation of variables to get $y(t) = (t + c)^3$ so that $y_2(t) = t^3$.
    - NOT UNIQUE!
    - The reason being that if we compute $\frac{\partial f}{\partial y} = 2y^{-\frac{1}{3}} = \frac{2}{y^{1/3}}$. Not continuous at $(t_0, y_0) = (0, 0)$! Can't use Exitence/Uniqueness theorem.
  - **Question2:** Take $\frac{dy}{dt} = 3y^{2/3}$ and $y(1) = 1$. Find where in the t-y plane solutions exist uniquely.
    - **Solution:** Unique solution exist uniquely in any rectangle not containing $(0, 0)$.

Applications of Uniqueness

- **Important:** The uniqueness condition says that if y_1, y_2 are two solutions to some ODE and y_1 and y_2 are equal at some point t_0. Then $y_1(t) = y_2(t)$ for all t in some interval.
- Its either all or nothing.
- Rephrase as this: "If two solutions to a 1st order ODE are ever in the same place at the same time, then they are the same function"!

Equilibrium Solutions:
Example4: (Equilibrium)

- Take $\frac{dy}{dt} = y(y-2)^2$.
  - Equilibrium solutions $y = 0, 2$. Then if iniital condition is $y(0) = 1$ then the uniqueness conditions show that $0 < y(t) < 2$, or else $y(t)$ would equal to the equilibrium solutions
  - Notice that $\frac{dy}{dt}(1) = +$ so that the slopes in between are positive. Draw a possible graph.

Example3: (Comparing solutions)

- Take $\frac{dy}{dt} = \frac{(1+t)^2}{(1+y)^2}$.
  - Easily check that $y_1(t) = t$ is solution.
  - Then if $y_2(t)$ is the unique solutions to the IVP

$$\frac{dy}{dt} = \frac{(1+t)^2}{(1+y)^2} \quad y(0) = -.1.$$

    - Really hard to solve
    - But then $y_2(t)$ can't cross the other solution $y_1(t) = t$.
    - So we can say that $y_2(t) < t$ for all t!!!!!!!! Draw a graph!

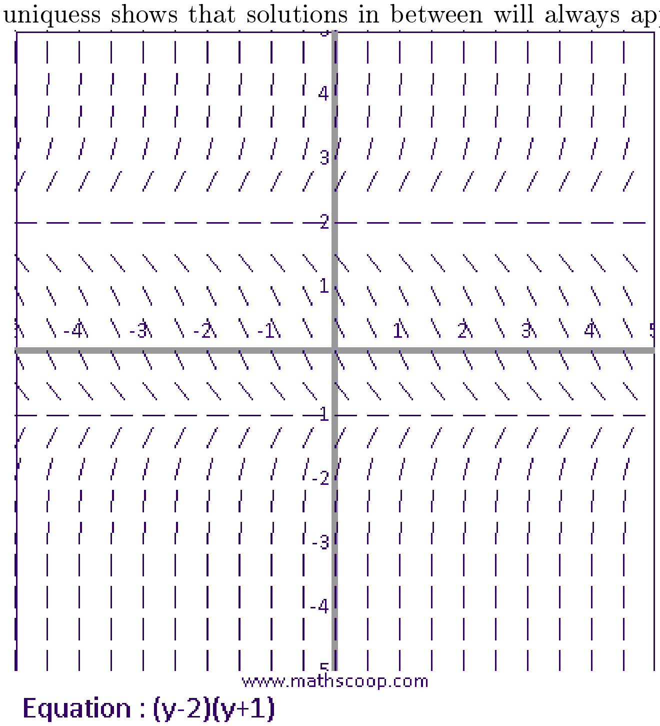
Example4: (Uniqueness and Qualitative Analysis)

- Take $\frac{dy}{dt} = (y-2)(y+1)$.
  - This equation is Autonomous so the equilibirum solutions are $y = 2, -1$. We can say:
    - Draw a graph. So by the uniqueness condition we know that solutions must stay between the equlibrium solutions.
    - They must stay the same sign, since these autonomous hence, the oritonal slabs are all the same.
    - Note that if $y(0) = 1$ then $\frac{dy}{dt}(0) = (-)$, which means $f(0) = -$. We know $f(y) < 0$ for all $y \in (-1, 2)$. Thus if we take $t \to \infty$ we have

$$\lim_{t\to\infty} \frac{dy}{dt} = \lim_{t\to\infty} f(t) = \lim_{t\to\infty} (-) = -$$

so it must keep getting decreasing and decreasing, so it $y(t) \to -1$.

- Draw graph a show that uniquess shows that solutions in between will always approach the equilibrium solutions.



Summary:

- We must check continuity conditions to have uniqueness and existence.
- Uniqueness implies, that solutions can't cross each other.
- Uniqueness implies, that solutions can't cross equilibrium solutions.
- We can use uniquess, to say that solutions are between other solutions.
  - Say $y_1(t) = t + 5$ and $y_2(t) = -t^2$ then uniquess shows
    - An IVP with $y(0) = 1$ must have $-t^2 < y(t) < t + 5$
    - $\lim_{t \to -\infty} y(t) = -\infty$
    - Draw a graph.

Sec. 2.5 - Autonomous Equations and Pupulation dynamics

- Recall that an **autonomous differential equation** is of the form

$$\frac{dy}{dt} = f(y).$$

  - We will only deal with autonomous for this section.
  - Autonomous are preferable for some physical models are autonomous (self-govering), For example a compressed spring the same amount has the same force at 4:00am and at 10:00pm.
- Here are some examples of autonomous equations:

Population growth/decay

- Assumption: The rate of growth of the population is proportional to the size of the population.

Thus if $k =$ proportionality constant (growth rate) we have

$$\frac{dP}{dt} = kP.$$

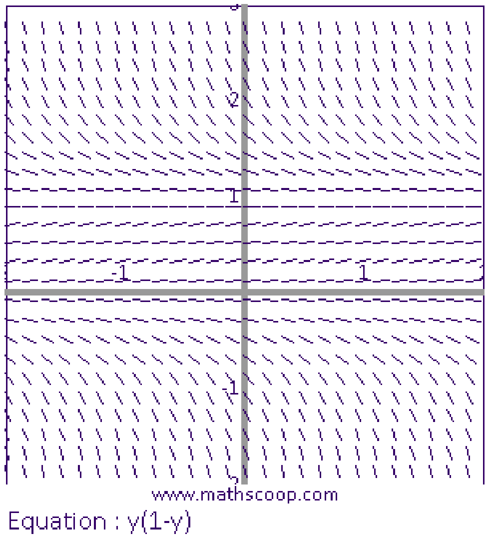
But here $P=$ dependent variable, t =time=independent variable. Thus $P = P(t)$ is actually a function! This is a ODE. We can also write it $P' = kP$, or the physics way, $\dot{P} = kP$.

- **Logistic Growth:**
- Assumption:
  - If population is small, then rate of growth is proportional to its size.
  - If population is to large to be supported by its resources and environment. Then the population will decrease, that $\frac{dP}{dt} < 0$.

  We can restate the assumptions as
  1. $\frac{dP}{dt} \approx kP$ if P is small.
  2. If $P > N$ then $\frac{dP}{dt} < 0$.
- In this case,

$$\frac{dP}{dt} = k\left(1 - \frac{P}{N},\right)P$$

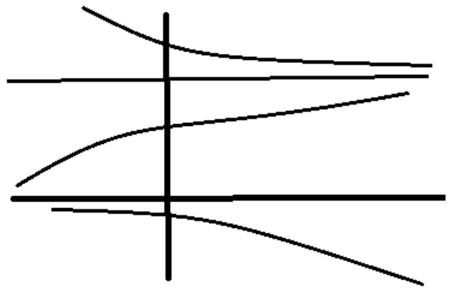
- **Another Example:** Suppose $\frac{dy}{dt} = y(1-y)$:



-
    - Since the slopes is the same at each horizontal direction we can compact this information to something *easier* to draw.
        * This will be called the phase line:
    - Rope Metaphor:
        * Start with IVP $\frac{dy}{dt} = f(y)$ and $y(0) = y_0$.
        * Draw a rope at start at y_0.
        * At each y write $f(y)$ on this rope to indicate the slope at that y.
            · **Directions:** If $f(y) = 0$ stay put, If $f(y) > 0$ then climb up the rope, if $f(y) < 0$ then climb down the rope
            · Bigger values for $f(y)$ means climb faster as t moves through time.
        * If you let $y(t)$ your location on the rope, then $y(t)$ is a solution to the IVP.

Phase Line:

- This rope is the Phase line, but instead of numbers we use arrows to represent the slope.
    - Draw Phase Line (2.equilibirium points, 3. arrows)
        * 
    - Use phase line to show that as t is close to $y = 1$ from below, then the function keeps increasing, and thus must approach assymtptically to the equilobrium solution.
    - Sketch a graph:



Sketching curves: (skip in class)
From our first sketch we can always notice the following things about sketching curves:

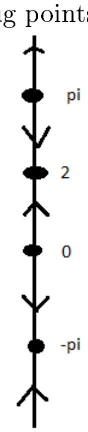
(1) If $f(y(0)) = 0$ then $y(0)$ is an equilibirum solution and $y(t) = y(0)$ for all t.
(2) If $f(y(0)) > 0$ then $y(t)$ is **increasing** for all t and either $y(y) \to \infty$ as $t \to \infty$ or $y(t)$ tends to first equilibirum point **larger** than $y(0)$.
(3) If $f(y(0)) < 0$ then then $y(t)$ is **decreasing** for all t and either $y(y) \to -\infty$ as $t \to \infty$ or $y(t)$ tends to first equilibirum point **smaller** than $y(0)$.

Example1 (Curve Sketching)

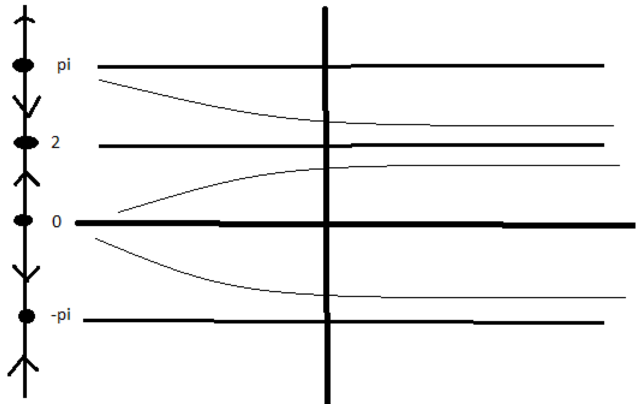
- We let

$$\frac{dy}{dt} = (2 - y)\sin y.$$

- Find equilirium points $y = 2$ and $y = n\pi$ (so infinite amount)
- Plug points and get that the phase line is :



-
- Talk about what happens when things are getting close to the equilibrium solutions.
- Sketch curve (with more **equilibriums** though)

- 

Example (We don't know how quickly things jump)

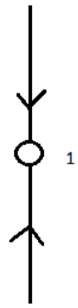
- Show that the graph $\frac{dP}{dt} = (1 - \frac{P}{20})^3(\frac{P}{5} - 1)P^7$ has Phase line $[\ominus 20 \oplus 5 \ominus 0 \oplus]$ $\begin{array}{c}\vee\\ 20\\ \wedge\\ 5\\ \vee\\ 0\\ \wedge\end{array}$ but 5 jumps to 20 very quikly
  - like 0.00001 quick.

Example (Not all solutions exists for all t, (asymptotes could exist))

- Take $\frac{dy}{dt} = (1 + y)^2$
  - Phase Line: $[\ominus - 1\oplus]$ $\begin{array}{c}\wedge\\ -1\\ \wedge\end{array}$ Sketch a curve:
  - These increasing/decreasing behavioe could be assymptotes. (Phase LINE DOES NOT TELL US THIS INFO)
  - ACTUAL SOL: $y(t) = -1 - \frac{1}{t+c}$. Asymptote at $t = c$.
    - ∗ If $y(0) > -1$ then draw possible curve.

Example (Cusps)

- Take $\frac{dy}{dt} = \frac{1}{1-y}$
  - Phase Line would be:



*

- **Sketch** cusp like curves:
- It has fallen into a hole once it reaches the dotted line.

Role of Equilibrium points:

- Solutions to Autonomous equations either
  - (1) Tend to $\pm\infty$
  - (2) Tend to the equilibrium solutions.
  - (3) stay the inreasing/decreasing within equil soltuion

Classification of Equilibrium Solutions

- Recall what **asymptotic** means.

(1) **Asymptotically Stable**
  (a) y_0 is a **Assymptotically stable** if any solution with initial condition sufficintely close to y_0 is asymptotic to y_0 as t incresase
  (b) Phase Line looks like this: $[\ominus y_0 \oplus]$ $\begin{matrix}\vee\\ y_0\\ \wedge\end{matrix}$
  (c) Graph looks like: (reminds you that it is faling into something)
  (d) In $f(y)$ vs. y graph we have $f'(y_0) < 0$.

(2) **Asymptotically Unstable:**
  (a) y_0 is a **Asymptotically unstable** if any solution with initial condition sufficintely close to y_0 tend torward y_0 as t decreases
  (b) Phase Line looks like this: $[\oplus y_0 \ominus]$ $\begin{matrix}\wedge\\ y_0\\ \vee\end{matrix}$
  (c) Graph looks like: (reminds you that it is coming from one place)
  (d) In $f(y)$ vs. y graph we have $f'(y_0) > 0$.

(3) **Semistable:**
  (a) y_0 is a **unstable** if it doesn't fit the category of a sink or source
  (b) Phase Line looks like this: $[\oplus y_0 \oplus]$ $\begin{matrix}\wedge\\ y_0\\ \wedge\end{matrix}$ or $[\ominus y_0 \ominus]$ $\begin{matrix}\vee\\ y_0\\ \vee\end{matrix}$
  (c) Graph looks like:

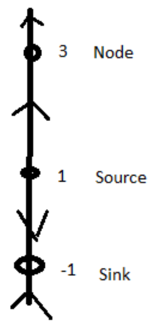
Example1: (Drawing solution from the $f(y)$ vs. y graph)

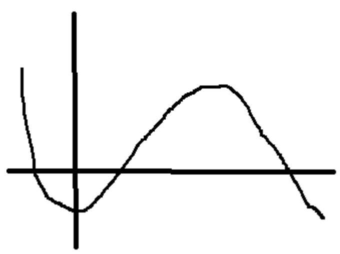
- $\frac{dy}{dt} = y^2 + y - 6 = (y+3)(y-2)$

- Phase Line [$\oplus 2 \ominus -3 \oplus$] $\begin{matrix} \wedge \\ 2 \\ \vee \\ -3 \\ \wedge \end{matrix}$
- Classify them!

Example2: (Using $f(y)$)

- We can figure out classification from just the graph of $f(y)$.



-
  - Here Node means semistable, Sink means stable, Source means unstable.
  - These are just different names for the same thing.
- **Example3:** Suppose we only know the graph of $f(y)$ not the actual formula.
  - 
- Then draw Phase line : [$\ominus c \oplus b \ominus a \oplus$] $\begin{matrix} \vee \\ c \\ \wedge \\ b \\ \vee \\ a \\ \wedge \end{matrix}$
- from this information and sketch curve.

Sec. 2.6 - Exact Equations

- Consider an equation
$$M(x,y)dx+N(x,y)dy=0,$$
we say this equation is **exact** if
$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$
  - **Example:** Suppose
$$\frac{dy}{dx}=\frac{-2x-y^2}{2xy}.$$
We can rewrite this as
$$\left(2x+y^2\right)dx+2xydy=0$$
then $M=2x+y^2$ and $N=2xy$. Computing the partial derivatives,
$$M_y=2y$$
$$N_x=2y$$
are $M_y=N_x$! Thus this equation is exact.

Theorem. *If M,N,M_y,N_x are all continuous and $Mdx+Ndy=0$ is exact then there exists a function ψ such that*
$$\psi_x(x,y)=M(x,y) \text{ and } \psi_y(x,y)=N(x,y)$$
and such that
$$\psi(x,y)=C,$$
gives an implicit solution to the ODE.

Proof. We only show if ψ satisfies $\psi_x=M$ and $\psi_y=N$ such that $\psi(x,y)=C$ defines a function $y=\phi(x)$ implicitely. Then we show $\phi(x)$ solves the ODE. Note that if
$$0=M(x,y)+N(x,y)y'=\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\frac{dy}{dx}=\frac{d}{dx}\left(\psi\left(x,\phi(x)\right)\right)$$
by the multivariable chain rule. Thus if we integrate both sides
$$\begin{aligned}0=\frac{d}{dx}\left(\psi\left(x,\phi(x)\right)\right) &\iff \int 0dx=\int\frac{d}{dx}\left(\psi\left(x,\phi(x)\right)\right)dx\\ &\iff c=\psi\left(x,\phi(x)\right),\end{aligned}$$
as needed. □

- **Solving exact equations**: If $Mdx+Ndy=0$ is exact then
$$\begin{cases}\psi_x=M(x,y) \implies & \psi=\int M(x,y)dx+h(y)\\ & \Downarrow\\ \psi_y=N(x,y) & =\psi_y=\frac{\partial}{\partial y}\left(\int M(x,y)dx\right)+h'(y)\end{cases}$$
and then solve for $h(y)$.
  - **Another way**: One may also solve it by starting with the second equation:
$$\begin{cases}\psi_x=M(x,y) & \psi_x=\frac{\partial}{\partial x}\left(\int N(x,y)dx\right)+g'(x)\\ & \Uparrow\\ \psi_y=N(x,y) \implies & =\psi=\int N(x,y)dy+g(x).\end{cases}$$

- **Example1:** We know

$$\left(2x+y^2\right)dx+2xydy=0$$

is exact.

  - **Step1:** Show it's exact(done earlier) and and complete the follow the arrows until you close the diagram:

$$\begin{array}{llcl} \text{Start here:} & \psi_x = 2x+y^2 & \Longrightarrow & \psi=\int\left(2x+y^2\right)dx+h(y) \\ & & & \boldsymbol{\psi = x^2+y^2x+h(y)} \\ & & & \Downarrow \\ & \psi_y = 2xy & \Longleftarrow & \psi_y = 2xy+h'(y) \end{array}$$

  - **Step2:** Solve for $h(y)$ by noting that since

$$\begin{aligned} 2xy = 2xy+h'(y) &\Longrightarrow h'(y)=0 \\ &\Longrightarrow h(y)=C. \end{aligned}$$

  - **Step3:** Put it all together and get

$$\psi(x,y)=x^2+y^2x+C$$

and hence the *implicit solution* is

$$x^2+y^2x=C.$$

- **Example2:** Solve

$$\left(y\cos x+2xe^y\right)+\left(\sin x+x^2e^y-y^2\right)y'=0.$$

  - **Step1:** To show it's exact note that

$$\left(y\cos x+2xe^y\right)dx+\left(\sin x+x^2e^y-y^2\right)dy=0,$$

and not hard to see that

$$\begin{aligned} M_y &= \cos x+2xe^y \\ N_x &= \cos x+2xe^y \end{aligned}$$

and they are equal, thus this ODE is exact. Follow the arrows until close the diagram:

$$\begin{array}{llcl} \text{Start here:} & \psi_x = y\cos x+2xe^y & \Longrightarrow & \psi=\int\left(y\cos x+2xe^y\right)dx+h(y) \\ & & & \boldsymbol{\psi = y\sin x+x^2e^y+h(y)} \\ & & & \Downarrow \\ & \psi_y = \sin x+x^2e^y-y^2 & \Longleftarrow & \psi_y = \sin x+x^2e^y+h'(y) \end{array}$$

  - **Step2:** Solve for $h(y)$ by noting that since

$$\begin{aligned} \sin x+x^2e^y-y^2=\sin x+x^2e^y+h'(y) &\Longrightarrow h'(y)=-y^2 \\ &\Longrightarrow h(y)=-\frac{y^3}{3} \end{aligned}$$

  - **Step3:** Put it all together and get

$$\psi(x,y)=y\sin x+x^2e^y-y$$

and hence the *implicit solution* is

$$y\sin x+x^2e^y-\frac{y^3}{3}=C.$$

- **Example3:** Find the value of b for which the given equation is exact, and then solve it using that b:

$$\left(xy^2 + bx^2y\right)dx + (x+y)\,x^2dy$$

  - **Step1:** If this equation is exact then $M_y = N_x$, but

$$M_y = 2xy + bx^2$$
$$N_x = 3x^2 + 2yx$$

  and are only equal when $b = 3$. Follow the arrows until close the diagram:

$$\begin{array}{lccc} \text{Start here:} & \psi_x = xy^2 + 3x^2y & \Longrightarrow & \psi = \int \left(xy^2 + 3x^2y\right)dx + h(y) \\ & & & \boldsymbol{\psi = \frac{1}{2}x^2y^2 + x^3y + h(y)} \\ & & & \Downarrow \\ & \psi_y = x^3 + x^2y & \Longleftarrow & \psi_y = x^2y + x^3 + h'(y) \end{array}$$

  - **Step2:** Solve for $h(y)$ by noting that since

$$\begin{aligned} x^3 + x^2y = x^2y + x^3 + h'(y) &\Longrightarrow h'(y) = 0 \\ &\Longrightarrow h(y) = C \end{aligned}$$

  - **Step3:** Put it all together and get

$$\psi(x,y) = \frac{1}{2}x^2y^2 + x^3y + C$$

  and hence the *implicit solution* is

$$\frac{1}{2}x^2y^2 + x^3y = C.$$

- **Example4 (advanced, if time permits):** Solve

$$\left(x\cos x + e^y\right)dx + xe^ydy$$

  - **Step1:** If this equation is exact then $M_y = N_x$, and

$$M_y = e^y$$
$$N_x = e^y$$

  Now note that it is eactualy easier to integrate N with respect to y: Thus we can start the diagram in the other direction

$$\begin{array}{lccc} & \psi_x = x\cos x + e^y & \Longleftarrow & = \psi_x = e^y + g'(x) \\ & & & \boldsymbol{\psi = xe^y + g(x)} \\ & & & \Uparrow \\ \text{Start here:} & \psi_y = xe^y & \Longrightarrow & \psi_y = \int \left(xe^y\right)dy + g(x) \end{array}$$

  - **Step2:** Solve for $g(x)$ by noting that since

$$x\cos x + e^y = e^y + g'(x) \Longrightarrow g'(x) = x\cos x$$

  but at the end of the day we can't avoid the harder integration, as we still need to integration by parts to

$$g(x) = x\sin x + \cos x$$

- **Step3:** Put it all together and get

$$\psi(x, y) = xe^y + x \sin x + \cos x$$

and hence the *implicit solution* is

$$xe^y + x \sin x + \cos x = C.$$

Sec. 2.7 - Euler's Method

- We explore a numerical technique to solving a differential equation.
- Suppose we are given an Initial Value problem

$$\frac{dy}{dt} = f(t, y) \qquad y(t_0) = y_0.$$

- The idea is to plot a slope field on top of it and use the slope field to take tiny steps dictated by the tangents on te slope field. Draw a picture (Start at initial point and drawing a line until we approach the next t_0, then pretend you already a slope at that point and continue. Do it the way they do it on DEtools) (Use Document camera)
- **Define:** h=step size. These are our $t-$axis increments.
- Define: $t_0 =$ is our starting point, then our next point will be $t_1 = t_0 + h$, then $t_2 = t_1 + h$.
  - For example suppose $t_0 = 1$ and $h = .5$, then $t_0 = 1, t_1 = 1.5, t_2 = 2, \ldots$.
- Draw a picture showing what the $y-$values would be.

How do we find the explicit values for y_k other than just guessing. Plot the points $(t_k y_k)$ and (t_{k+1}, y_{k+1}) on a graph and show that the secant must equal $f(t_k, y_k)$.

We know that

$$\frac{y_{k+1} - y_k}{t_{k+1} - t_k} = f(t_k, y_k)$$

so solve for y_{k+1} using the fact that $h = t_{k+1} - t_k$ and get

$$y_{k+1} = y_k + f(t_k, y_k) h.$$

Euler's Method:

Given an initial condition $y(t_0) = y_0$ and step size h, compute (t_{k+1}, y_{k+1}) from the preceding point (t_k, y_k) as follows:

$$\begin{aligned} t_{k+1} &= t_k + h \\ y_{k+1} &= y_k + f(t_k, y_k) h. \end{aligned}$$

Example:

Suppose we have the autonomous equation

$$\frac{dy}{dt} = 2y - 1 \qquad , y(0) = 1,$$

with $h = 0.1$ and $0 \leq t \leq 1$. Then

- Our first point is $(t_0, y_0) = (0, 1)$.
- We can compute the formula for this and get $t_{k+1} = t_k + .1$ and notice that $f(t, y) = 2y - 1$.

$$y_{k+1} = y_k + f(t_k, y_k) h = y_k + (2y_k - 1)(.1).$$

- Make a table:

k	t_k	$y_k = y_{k-1} + f(t_{k-1}, y_{k-1})\,h$	$f(t_k, y_k) = 2y_k - 1$
0	0	1	1
1	0.1	$y_1 = 1 + 1 \cdot (.1) = \mathbf{1.1}$	$f(t_1, y_1) = 2(1.1) - 1 = \mathbf{1.20}$
2	0.2	$y_2 = 1.1 + (1.20) \cdot (.1) = \mathbf{1.22}$	$f(t_1, y_1) = 2(1.22) - 1 = \mathbf{1.44}$
3	0.3	$y_3 = 1.22 + (1.20) \cdot (.1) = \mathbf{1.364}$	$f(t_1, y_1) = 2(1.22) - 1 = \mathbf{1.73}$
4	0.4	1.537	2.07
	.5	1.744	2.49
	.6	1.993	2.98
	.7	2.292	3.58
	.8	2.65	4.3
	0.9	3.080	5.16
	1.0	3.596	3.596

- Notice that actual value is $y(1) = \frac{e^2+1}{2} = 4.195$ and our approximation is $y(1) \approx 3.596$, which is a little short, but it makes sense all the slopes are always below the graph.

Example2:

Our previous example didn't have any $t's$ which requires more inputing of information. So suppose we have

$$\frac{dy}{dt} = -2ty^2, \quad y(0) = 1, \quad h = \frac{1}{2}$$

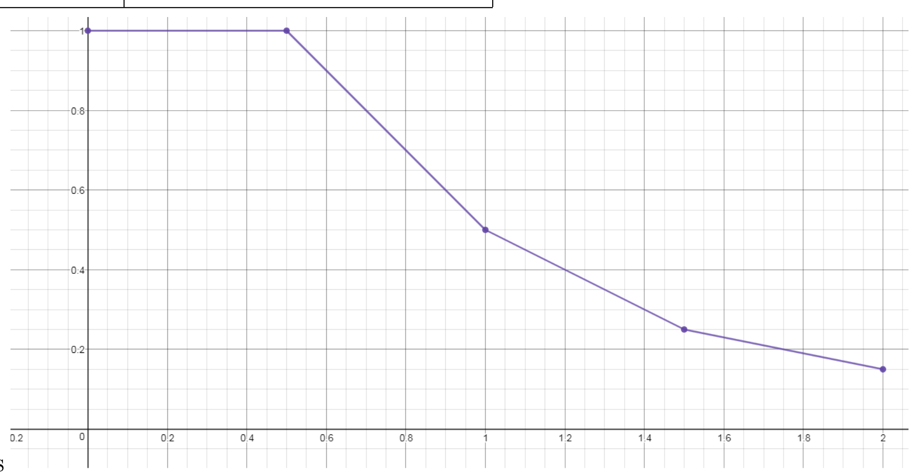
- Our first point is $(t_0, y_0) = (0, 1)$.
- We can compute the formula for this and get $t_{k+1} = t_k + .5$ and notice that $f(t, y) = -2ty^2$.

$$y_{k+1} = y_k + f(t_k, y_k)\,h = y_k + \left(-2t_k y_k^2\right)\left(\frac{1}{2}\right).$$

- Make a table:

k	t_k	$y_k = y_{k-1} + f(t_{k-1}, y_{k-1})\,h$	$f(t_k, y_k) = -2t_k y_k^2$
0	0	1	0
1	$\frac{1}{2}$	$y_1 = 1 + 0 \cdot (\frac{1}{2}) = \mathbf{1}$	$f(t_1, y_1) = -2\frac{1}{2}1^1 = -\mathbf{1}$
2	1	$y_2 = 1 + (-1) \cdot (\frac{1}{2}) = \mathbf{\frac{1}{2}}$	$f(t_1, y_1) = -2(1)(\frac{1}{2})^2 = -\mathbf{\frac{1}{2}}$
3	$1.5 = \frac{3}{2}$	$y_3 = \frac{1}{2} + (-\frac{1}{2}) \cdot (\frac{1}{2}) = \frac{1}{4}$	$f(t_1, y_1) = -2(\frac{3}{2})(\frac{1}{4})^2 = -\mathbf{\frac{3}{16}}$
4	2	$\frac{5}{32} = .09$	

- You can easily plot this as



- If you have extra time, log in the computer and show them DE tools and how it works.

Sec. 3.1 - 2nd Order Linear Equations - Homogeneous Eqs with Constant Coefficients

- A **general second order ODE** is of the form
$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right).$$
- A 2^{nd} **Order Linear ODE** is of the form
$$a(t)y'' + b(t)y' + c(t)y = d(t)$$
which can be rewritten as
$$y'' + p(t)y' + q(t)y = g(t).$$
- An **IVP** for a second order ODE needs to have two initial conditions:
$$y(t_0) = y_0,$$
$$y'(t_0) = y_0'.$$
- A 2nd Order ODE is called **Homogeneous** if
$$a(t)y'' + b(t)y' + c(t)y = 0$$
and **Nonhomogeneous** if
$$a(t)y'' + b(t)y' + c(t)y = d(t)$$
for some $d(t)$ that is identically zero.
- The first part of this Chapter we will focus on **2nd Order Linear homogeneous ODEs with constant coefficients**:
$$ay'' + by' + cy = 0$$
where a, b, c are real constants.
  - **Example:** Consider $y'' - y = 0$ or
$$y'' = y.$$
  - Can you think of a solution to this ODE from Calculus 1? A function where its second derivative is equal to itself?
    * Two Solutions: $y_1(t) = e^t$ and $y_2(t) = e^{-t}$.
    * But also not hard to check that c_1e^t and c_2e^{-t} are also solutions.
- **In General:** Consider the ODE
$$ay'' + by' + cy = 0.$$
  - Assume solutions are of the form $y(t) = e^{rt}$. Then
$$y(t) = e^{rt}$$
$$y'(t) = re^{rt}$$
$$y''(t) = r^2e^{rt},$$
and plugging this into the ODE we have
$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$
$$e^{rt}\left(ar^2 + br + c\right) = 0.$$
and since $e^{rt} \neq 0$ then
$$ar^2 + br + c = 0,$$
solve this EQ and get $r = r_1, r_2$.

 - This is called the **characteristic equation** of this ODE.
 - Assume the roots are **real and distinct**, then the **general solutions** is of the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

 * **Why though? We will justify this in the next section:**

Example1: Let's find the general solution of

$$y'' + 5y' + 6y = 0.$$

- **Step1:** We'll guess that the solution to a solution is $y(t) = e^{rt}$ for some r. Then get
$$\left(r^2 + 5r + 6\right) e^{rt} = 0$$
so that we must have $r^2 + 5r + 6 = (r+2)(r+3) = 0$ so that $r = -2, -3$.
- **Step2:** So $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$ are solutions and
$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$
is the general solution.
- .

Example2: Let's find the solution to the following IVP

$$y'' + 5' + 6y = 0 \quad y(0) = 2, y'(0) = -1.$$

- **Step1:** Solving for the particular solution. We have $y(0) = 2$ and $y'(0) = -1$. Differentiating $y(t) = c_1 e^{-2t} + c_2 e^{-3t}$ we get $y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$ and set up the following system:
$$\begin{aligned} c_1 + c_2 &= 2 \\ -2c_1 - 3c_2 &= -1 \end{aligned}$$
and get $c_1 = 5, c_2 = -3$. So the particular solution is
$$y(t) = 5e^{-2t} - 3e^{-3t}.$$

Example3: Let's find the general solution of

$$2\frac{d^2y}{dt^2} + 7\frac{dy}{dt} - 4y = 0.$$

- **Step1:** We'll guess that the solution to a solution is $y(t) = e^{rt}$ for some r. Then get
$$\left(2r^2 + 7r - 4\right) e^{rt} = 0$$
so that we must have $2r^2 + 7r - 4 = (2r-1)(r+4) = 0$ so that $r = \frac{1}{2}, -4$.
- **Step2:** So $y_1(t) = e^{t/2}$ and $y_2(t) = e^{-4t}$ are solutions and
$$y(t) = c_1 e^{t/2} + c_2 e^{-4t}$$
is the general solution.

Sec. 3.2 - Solutions to Linear Equations; the Wronskian

- In this section, we will consider equations of the form

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

  where a, b, c are constants.
- This is a second order, linear, constant coefficient,**homegeneous** equation.
- Our **goal** is again to find the general solution of these equations.

Theorem. *(Existence and Uniqueness) Consider the IVP*

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

where p, q, g are continuous on an open interval I that contains t_0. Then there exists a unique solution $y = \phi(t)$, and the solution exists throughout all of I.

- Recall that this theorem implies solutions to this IVP
  - 1. exist
  - 2. is unique
  - 3. the solution ϕ is defined throughout all of I. In fact it says more, ϕ is at least twice differentiable on I.
- **Example1:** Find the longest interval in which the solution to the IVP is certain to exists:

$$\left(t^4 - 4t^2\right) y'' + \cos t y' - e^t y = 0, \quad y(1) = 2, \ \ y'(1) = 1.$$

- **Solution:** Rewriting as

$$y'' + \frac{\cos t}{t^2\left(t^2-4\right)}y' - \frac{e^t}{t^2\left(t^2-4\right)}y = 0.$$

  so that $p(t) = \frac{\cos t}{t^2(t^2-4)}$ and $q(t) = -\frac{e^t}{t^2(t^2-4)}$ which are both continious on $(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)$. Since $t_0 = 1 \in (0, 2)$ then $I = (0, 2)$ is the longest interval where $p(t)$ and $q(t)$ are both continuous that contains t_0.
- **Fact:** (**Principle of Superposition**) If y_1 and y_2 are two solutions to an ODE

$$y'' + p(t)y' + q(t)y = 0,$$

  then the linear combination $y(t) = c_1y_1(t) + c_2y_2(t)$ is also a solution for any values c_1, c_2.
  - **Warning:** This only works if equation is linear and homoheneous.
  - **Summarizing the Principle:** Combining solutions gives another solution.
  - **Example1:** Suppose $y_1(t) = e^{-t}$ and $y_2(t) = e^t$ are two solutions to $y'' - y = 0$. Since this is a linear homogeneous ODE then the principle of superposition says that the function

$$y(t) = 2e^{-t} + 3e^t$$

    is also a solution.
  - **Example2:** It is not hard to check that $y_1(t) = 1$ and $y_2(t) = t^{\frac{1}{2}}$ are solutions to

$$yy'' + \left(y'\right)^2 = 0, \ \ t > 0.$$

    * **Part (a):** Show $y(t) = 1 + 2t^{\frac{1}{2}}$ is not a solution to this ODE:

· **Solution:** First compute

$$y(t) = 1 + 2t^{\frac{1}{2}}$$
$$y'(t) = t^{-\frac{1}{2}}$$
$$y''(t) = -\frac{1}{2}t^{-\frac{3}{2}}$$

To show this simply check if the LHS equal to 0:

$$LHS = yy'' + (y')^2 = \left(1 + 2t^{\frac{1}{2}}\right)\left(-\frac{1}{2t^{3/2}}\right) + \left(\frac{1}{t^{\frac{1}{2}}}\right)^2$$
$$= -\frac{1}{2t^{3/2}} - \frac{1}{t} + \frac{1}{t} = -\frac{1}{2t^{3/2}} \neq 0,$$

thus it isn't a solution.

∗ **Part (b):** Why does this not contradict the Principle of Superposition?

· **Solution:** To apply the principle the equation needs to be linear, the term $(y')^2$ in the ODE makes this nonlinear, hence we can't even use the principle in the first place.

- **Question we want to answer in this section:** Suppose $y_1(t)$ and $y_2(t)$ are two solutions to a linear homogeneous equation. When do we know that

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is the **general solution** to the ODE? Meaning when do we know that we can obtain **every single solution** to an IVP? To answer that we need to define a couple of things.

- **Definition:** The determinant of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

- **Definition:** The **Wronskian** of the solutions $y_1(t)$ and $y_2(t)$ to a linear homogeneous ODE

$$W = W(y_1, y_2) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

Theorem. ***(General Solution Theorem)*** *Suppose y_1 and y_2 are two solutions to the ODE*

$$y'' + p(t)y' + q(t)y = 0.$$

Then the family of solutions

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for arbtitrary c_1, c_2 is the general solution (meaning includes every solution to the ODE) if and only if the Wronskian $W(y_1, y_2)$ is not zero for at least one point.

- **Example1:** Find the general solution to

$$y'' + 4y' - 5y = 0.$$

  – **Solution:** In the last section we showed that to find solutions to this ODE we simply need to solve the characteristic equation

$$r^2 + 4r - 5 = (r - 1)(r + 5) = 0$$

and get $r = 1, -5$ so that

$$y(t) = c_1 e^t + c_2 e^{-5t}$$

gives other solutions to the ODE. To show this gives all of them, we simply need to show the Wronksian is not always zero:

$$\begin{aligned} W\left(e^t, e^{-5t}\right) &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^t & e^{-5t} \\ e^t & -5e^{-5t} \end{vmatrix} \\ &= -5e^{-4t} - e^{-4t} \\ &= -6e^{-4t} \\ &\neq 0. \end{aligned}$$

- **Restate the theorem:** To find the general solution, we only need to find two (y_1, y_2) solutions whose Wronskian is nonzero.
  - This means from 2 *fundamental* solutions, we can find all others.
  - This is why we define the following termilogy
- **Definition:** The solutions y_1 and y_2 are said to form **a Fundamental set of solutions** to

$$y'' + p(t)y' + q(t)y = 0$$

if $W(y_1, y_2) \neq 0$.

- **Example1:** Verify that $y_1(t) = t^{\frac{1}{2}}$ and $y_2(t) = t^{-1}$ form a fundamental set of solutions of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0.$$

  - **Solution:**
  - **Part(a):** First we verify these are indeed solutions by plugging them into the LHS and checking that they equal zero. First computer some derivatives

$$\begin{array}{rr} y_1(t) = t^{\frac{1}{2}} & y_2(t) = t^{-1} \\ y_2'(t) = \frac{1}{2}t^{-\frac{1}{2}} & y_2'(t) = -t^{-2} \\ y_3'(t) = -\frac{1}{4}t^{-\frac{3}{2}} & y_2''(t) = -t^{-2}. \end{array}$$

  Plugging y_1 into LHS we get

$$\begin{aligned} LHS &= 2t^2 y_1'' + 3ty_1' - y_1 \\ &= 2t^2\left(-\frac{1}{4}t^{-\frac{3}{2}}\right) + 3t\left(\frac{1}{2}t^{-\frac{1}{2}}\right) - \left(t^{\frac{1}{2}}\right) \\ &= -\frac{1}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} \\ &= 0. \end{aligned}$$

  Thus y_1 is a solution. It is very similar to show y_2 is a solution.
  - **Part (b):** To show $y_1 y_2$ form a fundamental set of solutions, we simply need to show that $W(y_1, y_2)$ is nonzero:

$$W(y_1, y_2) = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2} \neq 0$$

  - which is nonzero for $t > 0$.
- **Question:** Given an ODE, when do you know there exists a fundamental set of solutions?
  - **Fact:** Assume p, q are continuous and find a solutions y_1, y_2 with different values of their initial conditions. Then they form a fundamental set of solutions.

Sec. 3.3 - Complex Roots of the Characteristic Equation

Complex numbers:

- Here are some facts. Complex numbers are of the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.
  - $i^2 = -1$. Remember this.
- Also we'll need to know **Euler's Formula:** $e^{ib} = \cos b + i \sin b$. So

$$e^{a+ib} = e^a e^{ib} = e^a (\cos b + i \sin b) = e^a \cos b + ie^a \sin b.$$

- **Complex roots to the Char. Eq.**
  - Suppose we are solving

$$ay'' + by' + cy = 0$$

and we solve the characteristic equation

$$ar^2 + br + c = 0$$

and get that the roots are

$$r = \lambda + i\mu \text{ and } r = \lambda - i\mu.$$

Remenber that complex roots always come in conjugate pairs.

  - Choosing the first root $r = \lambda + i\mu$ then (just like the previous section) one solution is of the form

$$\begin{aligned} y(t) &= e^{rt} = e^{(\lambda+i\mu)t} = e^{\lambda t} e^{i\mu t} \\ &= e^{\lambda t} (\cos(\lambda t) + i \sin(\lambda t)) \\ &= e^{\lambda t} \cos(\lambda t) + ie^{\lambda t} \sin(\lambda t) \\ &= u(t) + iv(t) \end{aligned}$$

where $u(t) = e^{\lambda t} \cos(\lambda t)$ is the real part and $v(t) = e^{\lambda t} \sin(\lambda t)$ is the imaginary part.

    - But this is an **imaginary solution! We like real solutions!**
    - The following theorem will help us!

- **Theorem:** If $y(t) = u(t) + iv(t)$ is a complex solution to an ODE of the form $ay'' + by' + cy = 0$. Then so are $u(t)$ and $v(t)$!
  - **What does this imply?** Therefore since $u(t) = e^{\lambda t} \cos(\lambda t)$ and $v(t) = e^{\lambda t} \sin(\lambda t)$ are solutions we can compute (after some tedious work) that the Wronskian of u and v are:

$$W(u, v)(t) = \mu e^{2\lambda t} \neq 0 \text{ as long as } \mu \neq 0.$$

Hence by the General Solution Theorem (from last section), because the Wronskian is not zero then $u(t)$ and $v(t)$ form a fundamental set of solutions. Meaning their linear combination gives us the general solution!

Summary of Shortcuts:

So if you have

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

then find the roots of

$$ar^2 + br + c = 0.$$

The general solutions are the following:

roots:	The general Solution	Example
r_1, r_2 =real, distint	$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$(r+1)(r-1) = r^2 - 1 = 0$
$r = \lambda \pm i\mu$,imaginary	$y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$	$r^2 + 1 = 0$

Example1: Let's find the general solution of

$$\frac{d^2 y}{dt} + 4\frac{dy}{dt} + 13y = 0$$

- **Step1:** We can jump straight to the characteristic equation:

$$r^2 + 4r + 13 = 0$$

and completiting the square trick (add/subtract $(b/2)^2$) we get $r = -2 \pm 3i$.

- **Step2:** The general solution is

$$y(t) = c_1 e^{-2t} \cos 3t + c_2 e^{-2t} \sin 3t.$$

Example2: Let's find the particular solution to the IVP:

$$y'' + 4y = 0, \quad y(0) = -2, \;\; y'(0) = 1$$

- **Step1:** We can jump straight to the characteristic equation:

$$r^2 + 9 = 0$$

and get $r = \pm 3i$

- **Step2:** The general solution is

$$\begin{aligned} y(t) &= c_1 e^{0t} \cos 3t + c_2 e^{0t} \sin 3t. \\ &= c_1 \cos 3t + c_2 \sin 3t. \end{aligned}$$

- **Step3:** Using the initial conditions $y(0) = 0, \;\; y'(0) = 1$ we need to first take a derivative

$$\begin{aligned} y(t) &= c_1 \cos 3t + c_2 \sin 3t \\ y'(t) &= -3c_1 \sin 3t + 3c_2 \cos 3t \end{aligned}$$

hence

$$\begin{aligned} -2 &= y(0) = c_1 + 0 \\ 1 &= y'(0) = 0 + 3c_2 \end{aligned}$$

so that

$$c_1 = -2, c_2 = \frac{1}{3}.$$

hence the solution is

$$y(t) = -2\cos 3t + \frac{1}{3}\sin 3t.$$

- **Example3:** Suppose we get that the general solution comes out to

$$y(t) = c_1 e^{3t} \cos t + c_2 e^{3t} \sin t.$$

Then just remenber that you need to use product rule to find the derivative of $y(t)$:

$$y'(t) = 3c_1 e^{3t} \cos t - c_1 e^{3t} \sin t + 3c_2 e^{3t} \sin t + c_2 e^{3t} \cos t.$$

SEC. 3.4 - REPEATED ROOTS; REDUCTION OF ORDER

- Suppose we have
$$ay'' + by' + cy = 0$$
and we only have one root $r = r_1$ to the characteristic equation $ar^2 + br + c = 0$. Then we only have one solution $y(t) = c_1 e^{r_1 t}$. We need a second different solution to get the general solution.
- **The Method of Reduction of Order**
- **Example1:** Consider
$$y'' + 6y' + 9y = 0.$$
  - Then the characteristic equation is $r^2 + 6r + 9 = (r+3)^2 = 0$, gives the repeated root of $r_1 = r_2 = -3$. Hence we only have one solution so far:
$$y_1(t) = c_1 e^{-3t}$$
but we need another solution $y_2(t)$ that is not a multiple of y_1.
  - The **Method of Reduction** is to guess the solution is of the form
$$y = v(t)y_1(t),$$
$$y = v(t)e^{-3t}.$$
  - **Solution:**
  - **Step1:** After making guess. We need to simply put our guess into the ODE and find out what $v(t)$ is! First let's take some derivatives:
$$\begin{aligned} y &= v(t)e^{-3t} \\ y' &= v'(t)e^{-3t} - 3v(t)e^{-3t} \\ y'' &= v''(t)e^{-3t} - 3v'(t)e^{-3t} - 3v'(t)e^{-3t} + 9v(t)e^{-3t} \\ &= v''(t)e^{-3t} - 6v'(t)e^{-3t} + 9v(t)e^{-3t}. \end{aligned}$$
  - **Step2:** Now plug this into the LHS of the ODE: $y'' + 6y' + 9y = 0$, like this
$$\begin{aligned} LHS = y'' + 6y' + 9y &= v''(t)e^{-3t} - 6v'(t)e^{-3t} + 9v(t)e^{-3t} \\ &\quad + 6\left(v'(t)e^{-3t} - 3v(t)e^{-3t}\right) \\ &\quad + 9\left(v(t)e^{-3t}\right) \\ &= \text{do simplification} \\ &= v''(t)e^{-3t} \end{aligned}$$
Since $e^{-3t} \neq 0$, therefore
$$\begin{aligned} v''(t) = 0 &\implies v'(t) = c_1 \\ &\implies v(t) = c_1 t + c_2 \end{aligned}$$
We got it!
  - **Step3:** The general solution must be of the form
$$y(t) = (c_1 t + c_2)\, e^{-3t},$$
$$\boldsymbol{y(t) = c_1 t e^{-3t} + c_2 e^{-3t}}.$$

- **Side note:** To check that $y_1(t) = e^{-3t}$ and $y_2(t) = te^{-3t}$ really do give the **General solution** we simply need to check the *Wronskian* is not zero: N

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & e^{-3t} - 3te^{-3t} \end{vmatrix} = e^{-6t} \neq 0.$$

  by the *general solution theorem* we know that $y = c_1y_1 + c_2y_2$ is the general solution!

- This method works in general for repeated roots. We summarize it here:

Final Summary of Shortcuts:

So if you have

$$ay'' + by' + cy = 0$$

then find the roots of

$$ar^2 + br + c = 0.$$

The general solutions are the following:

roots:	The general Solution	Example
r_1, r_2 =real, distint	$y(t) = c_1e^{r_1t} + c_2e^{r_2t}$	$(r+1)(r-1) = r^2 - 1 = 0$
$r = \lambda \pm i\mu$,imaginary	$y(t) = c_1e^{\lambda t}\cos\mu t + c_2e^{\lambda t}\sin\mu t$	$r^2 + 1 = 0$
$r = r_1$,real, repeasted	$y(t) = c_1e^{r_1t} + c_2te^{r_1t}$.	$(r-2)^2 = 0$

- **Example2:** Find the general solution of

$$y'' - 10y' + 25y = 0.$$

  - **Solution:** Note that the characteristi equation is $r^2 - 10r + 25 = (r-5)^2 = 0$ so that we have a repeated root $r = 5$. Hence the general solution is

$$y(t) = c_1e^{5t} + c_2te^{5t}.$$

- **Example3:**(More on Method of Reduction) Suppose we know that $y_1(t) = t$ is a solution to

$$t^2y'' + 2ty' - 2y = 0, \quad t > 0.$$

  Find the second solution $y_2(t)$ of this ODE.
  - **Solution:**
  - **Step1:** Recall we guess

$$y_2(t) = v(t)y_1(t) = v(t)t$$

  and we are going to figure out what $v(t)$ is supposed to be. Take derivatives:

$$\begin{aligned} y_2(t) &= v(t)t \\ y_2'(t) &= v'(t)t + v(t) \\ y_2''(t) &= v''(t)t + v'(t) + v'(t) \\ &= v''(t)t + 2v'(t). \end{aligned}$$

- **Step2:** Plug y_2 and its derivatives into the LHS of the ODE:
$$\begin{aligned} LHS = t^2y_2'' + 2ty_2' - 2y_2 &= t^2\left(v''(t)t + 2v'(t)\right) \\ &+ 2t\left(v'(t)t + v(t)\right) \\ &- 2\left(v(t)t\right) \\ &= \text{simplify} \\ &= t^3v''(t) + 2t^2v'(t) \\ &+ 2t^2v'(t) + 2tv(t) \\ &- 2tv(t) \\ &= t^3v'' + 4t^2v' = 0 \end{aligned}$$
and setting equal to zero means
$$v''t + 4v' = 0.$$
Step3: Solving the ODE $t^3v'' + 4t^2v' = 0$ we do the **substitution** $w = v'$ (**sub-technique:** when $a(t)v'' + b(t)v' = 0$ use substitution $w = v'$) and get
$$w' + \frac{4}{t}w = 0$$
hence we can use integrating factors to get $\mu(t) = e^{\int \frac{4}{t}dt} = e^{4\ln t} = t^4$ hence the solution is
$$\begin{aligned} w(t) &= \frac{1}{t^4}\left[\int t^4 \cdot 0dt + C\right] \\ &= \frac{c_1}{t^4} \end{aligned}$$
Thus
$$v' = w = c_1t^{-4}$$
hence
$$v = c_1t^{-3} + c_2.$$
- **Step4:** To finish we have that $y_2 = v \cdot t = \left(c_1t^{-3} + c_2\right)t = c_1t^{-2} + c_2t$. Since $y_1(t) = t$ and we want a different solution we can make $c_2 = 0$ and $c_1 = 1$. Thus
$$y_2(t) = t^{-2}$$
is a different solution.

Sec. 3.5 - non-Homogeneous - Method of undetermined Coefficients

- Consider the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \quad \text{(non-hom)}$$

where p, q, g are (continuous) functions on some open interval I. Consider the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \quad \text{(hom)}.$$

whose general solution we'll call y_c.

Theorem 1. ***(General Solution for non-hom EQs)*** *The general solution of the* ***Non-homogeneous EQ*** *above is given by*

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t)$$

where y_1, y_2 are a fundamental set of solutions of the corresponding ***Homogeneous Equation****, and $y_p(t)$ is a particular solution to the* ***Non-homogeneous equation.***

- Steps to solving $y'' + p(t)y' + q(t)y = g(t)$
  - **Step1:** We already know how to find the fundamental set of solutions y_1, y_2 for the homogeneous equation. We have that $y_c = c_1y_1 + c_2y_2$ is the gen solution to the *corresponding* homogeneous equation.
  - **Step2:** Find a particular solution y_p using the **method of undertermined coefficients**. (I'll show this in a minute. It's a bit compliated but we'll work it out step by step)
  - **Step3:** The general solution is when you add them together: $y(t) = y_c + y_p = c_1y_1 + c_2y_2 + y_p$.

The Method of Undetermined Coefficients (MOUC):

- **Main Idea:** The idea of MOUC is to guess what the solution y_p based on what $g(t)$ looks like.
  - Our guess of y_p will always be the general form of $g(t)$.
  - The following chart explains that if you see $g(t)$ as in the Left column, then your guess will be in the right column:

If $g(t)$ looks like	Then $y_p(t)$ is
$P_n(t) = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_0$	$t^s\left[A_mt^m + A_{m-1}t^{m-1} + \cdots + A_0\right]$
$e^{\alpha t}P_m(t)$	$t^se^{\alpha t}\left[A_mt^m + A_{m-1}t^{m-1} + \cdots + A_0\right]$
$P_m(t)e^{\alpha t}\cos\beta t$ or $P_m(t)e^{\alpha t}\sin\beta t$	$t^s\left[(A_mt^m + \cdots + A_0)e^{\alpha t}\cos\beta t + (B_mt^m + \cdots + B_0)e^{\alpha t}\sin\beta t\right]$

- Here $s =$ the smallest nonnegative integer ($s = 0, 1,$or 2) such that no term of y_p is a solution to the corresponding homogeneous equation. Meaning we don't want repeats!

Example1: Find the solution to the following IVP:

$$y'' + 5y' + 6y = e^{-t}. \quad y(0) = 1, y'(0) = \frac{1}{2}$$

- **Step1:** Find $y_c(t)$, which is simply the general solution of

$$y'' + 5y' + 6y = 0$$

but we learned that we must solve the charateristic polynomial $r^2 + 5r + 6 = (r+2)(r+3) =$ and get $r = -2, -3$ so that the solution is

$$y_c(t) = c_1e^{-2t} + c_2e^{-3t}.$$

- **Step2:** We find $y_p(t)$ by making our guess and to find the undertermined coefficient. So we let $y_p(t) = Ae^{-t}$ and plug y_p into the LHS:

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= Ae^{-t} - 5Ae^{-t} + 6Ae^{-t} \\ &= 2Ae^{-t} \end{aligned}$$

- **Step3:** Set the LHS equal to the RHS and solve for A to get

$$2Ae^{-t} = e^{-t}$$

so that $A = \frac{1}{2}$.
- **Step4:** Plug A back in and get $y_p(t) = \frac{1}{2}e^{-t}$ and a general solution of

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{2}e^{-t}.$$

- **Final IVP Step:** Now we need to find c_1 and c_2 using $y(0) = 1$ and $y'(0) = \frac{1}{2}$ and set up the following system of equations:

$$\begin{aligned} c_1 + c_2 + \frac{1}{2} &= 1 \\ -2c_1 - 3c_2 - \frac{1}{2} &= \frac{1}{2} \end{aligned}$$

which comes from $y(t) = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{2}e^{-t}$ and $y'(t) = -2k_1 e^{-2t} - 3k_2 e^{-3t} - \frac{1}{2}e^{-t}$. Solving this we get $c_1 = \frac{5}{2}$ and $c_2 = -2$ thus the solution to the IVP is

$$y(t) = \frac{5}{2}e^{-2t} - 2e^{-3t} + \frac{1}{2}e^{-t}.$$

Example2: Find the general solution of

$$\frac{d^2y}{dt} - 5\frac{dy}{dt} + 4y = e^{4t}.$$

- **Step1:** Find $y_c(t)$, which is simply the general solution of

$$\frac{d^2y}{dt} - 5\frac{dy}{dt} + 4y = 0$$

but we solve $r^2 - 5r + 4 = (r-1)(r-4) = 0$ and get $r = 1, 4$ so that the solution is

$$y_c(t) = c_1 e^t + c_2 e^{4t}.$$

- **Step2:** Wrong Guess $y_p(t) = Ae^{4t}$ because

$$\frac{d^2y_p}{dt} - 5\frac{dy_p}{dt} + 4y_p = 16Ae^{4t} - 20Ae^{4t} + 4Ae^{4t} = 0.$$

But we should have known that this wouldn't work. Because he term e^{4t} is part of the homogeneous solution then plugging it the LHS will of course give us zero. Thus whenever you see this second guess by multiplying by t.
  - **Second Guess** should ways be $y_p(t) = Ate^{4t}$. Find y_p' and $y_p''(t)$ on the side and plug into LHS and get

$$\begin{aligned} \frac{d^2y_p}{dt} - 5\frac{dy_p}{dt} + 4y_p &= \left(8Ae^{4t} + 16Ate^{4t}\right) - 5\left(Ae^{4t} + 4Ate^{4t}\right) + 4Ate^{4t} \\ &= 3Ae^{4t} \end{aligned}$$

  - Set LHS equal to RHS and get $3Ae^{4t} = e^{4t}$ so that $A = \frac{1}{3}$.
- **Step3:** Plug A back in and get $y_p(t) = \frac{1}{3}e^{4t}$ and a general solution of

$$y(t) = c_1 e^t + c_2 e^{4t} + \frac{1}{3} t e^{4t}.$$

Example3: Find the general solution of

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 4\cos 2t$$

- **Step1:** Find y_c which is the general solution to the unforced equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 10y = 0$$

which since $r^2 + 2r + 10 = 0$ gives $r = -1 \pm 3i$ must be

$$y_c(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t.$$

- **Step2:** Now as long as the RHS $g(t)$ is not part of y_c then we can use that as our guess. So we let $y_p(t) = A\cos 2t + B\sin 2t$.
- **Step3:** Plug into the LHS and set equal to RHS

$$\begin{aligned} \frac{d^2y_p}{dt^2} + 2\frac{dy_p}{dt} + 10y &= [-4A\cos 2t - 4B\sin 2t] \\ &\quad +2[-2A\sin 2t + 2B\cos 2t] + 10[A\cos 2t + B\sin 2t] \end{aligned}$$

which gives us

$$\text{LHS} = [-4A + 4B + 10A]\cos 2t + [-4B - 4A + 10B]\sin 2t = 4\cos 2t + 0\cdot\sin 2t = \text{RHS}$$

so that

$$\begin{aligned} 6A + 4B &= 4 \\ -4A + 6B &= 0 \end{aligned}$$

gives us $A = \frac{6}{13}, B = \frac{4}{13}$.
- **Step4:** Plug into general solution of $y(t) = y_c(t) + y_p(t)$ and get

$$y(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t + \frac{6}{13}\cos 2t + \frac{4}{13}\sin 2t.$$

Example4: Find the general form of a particular solution of

$$y'' - 2y' - 3y = 5te^{-t}.$$

- **Step1:** Find y_c which is $y_c = c_1 e^{-t} + c_2 e^{3t}$.
- **Step2:** Using our table our first guess will be

$$y_p = (At + B)e^{-t}$$

since $At + B$ is the general form of a one degree polynomial. But this doesn't work because Be^{-t} is included in the y_c as $c_1 e^{-t}$
- **Second Guess:**

$$y_p = t(At + B)e^{-t}$$

now both $At^2 e^{-t}$ and Bte^{-t} are different than the terms in y_c. Thus this is our correct guess.

Example5: Find the general form of a particular solution of

$$y'' + 6y' + 9y = -7te^{-3t} + t^3$$

- **Step1:** Then the characteristic equation is $r^2 + 6r + 9 = (r+3)^2 = 0$, gives the repeated root of $r_1 = r_2 = -3$. Hence
$$y_c(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$
- **Step2:** Using our table we make our **first guess** as
$$y_p = (At + B)\, e^{-3t} + Ct^3 + Dt^2 + Et + F$$
but this is wrong since $(At + B)\, e^{-3t}$ is included in the y_c. So our **second guess** is to multiply *only that part* by t, and get
$$y_p = t\,(At + B)\, e^{-3t} + Ct^3 + Dt^2 + Et + F$$
but this still doesn't work since Bte^{-3t} in included in the y_c as $c_2 t e^{-3t}$.
  - Our **Third guess** is to multiply again only that part by t and get
$$y_p = t^2\,(At + B)\, e^{-3t} + Ct^3 + Dt^2 + Et + F$$
this works since none of the terms in the y_p are included in the y_c.
- **Example6 (past exam question):** Find the general form of a particular solution of
$$y'' + y = t + t\sin t$$
- **Step1:** As in Example 3 we know $y_c(t) = c_1 \cos t + c_2 \sin t$.
- **Step2:** Our first guess would normally be $y_p = At + B + [(Dt + E)\cos t + (Ft + G)\sin t]$ but notice that since $E\cos t$ and $G \sin t$ is included in the y_c we need to muiltiply by t and get our final guess of
$$y_p = At + B + t\,[(Dt + E)\cos t + (Ft + G)\sin t]$$

Example7: Find the general form of a particular solution of
$$y'' + 2y' + 10y = 4e^{-t}\cos 3t + 17$$
- **Step1:** As in Example 3 we know $y_c(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t$.
- **Step2:** Since $e^{-t}\cos 3t$ is already inside our y_c we need to multiply by t .
$$y_p = t\left(Ae^{-t}\cos 3t + Be^{-t}\sin 3t\right) + C.$$
Note that 17 is a zero degree polynomiall , which is why we have the C in the y_p.

Sec. 3.6 -Variation of Parameters

- Consider the equation
$$y'' + 4y = 3\csc t = \frac{3}{\sin t}$$
  - MOUC doesn't work with quotients, only products.
- We will learn a general formula to solving more general linear non-homoheneous 2nd order ODEs

Theorem. *(Variation of Parameters) If p, q, and g are continuous on an open interval I, and if the functions $\{y_1, y_2\}$ form a fundamental set of solutions to the corresponding homogeneous EQ*
$$y'' + p(t)y' + q(t)y = 0.$$
Then a particular solution to
$$y'' + p(t)y' + q(t)y = g(t),$$
is given by
$$\begin{aligned} y_p(t) &= -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}ds \\ &= -y_1(t)\left[\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)}dt\right] + y_2(t)\left[\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}dt\right], \text{ if an antiderivative exists} \end{aligned}$$
where t_0 is any value in I. Then general solution to the non-homogeneous solution is
$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t).$$

Proof. See proof in the book. Or see Example 1 in the book for an explaination for this method. But the idea is this: Suppose
$$y_c(t) = c_1y_1(t) + c_2y_1(t)$$
is the general solution to
$$y'' + p(t)y' + q(t)y = 0.$$
Then the idea is to use the following guess:
$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$
for non-homogeneous equation. and also make the extra assumption that
$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0. \quad (\star)$$
Then take derivatives, simplify and put them back into ODE. Which will always reduce to
$$\begin{aligned} LHS &= y_p'' + p(t)y_p' + q(t)y_p \\ &= \text{work} \\ &= u_1'y_1'(t) + u_2'(t)y_2'(t) \end{aligned}$$
and set LHS to RHS which is $g(t)$ hence we get equation, that is
$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t). \quad (\star\star)$$
Putting $(\star)$ and $(\star\star)$ together we have the two equations:
$$\begin{cases} u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0 \\ u_1'y_1'(t) + u_2'(t)y_2'(t) = g(t) \end{cases}$$

which boils to solving for $u_1'(t)$ and $u_2'(t)$ and getting

$$\begin{cases} u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1,y_2)(t)} \\ u_2'(t) = \frac{y_1(t)g(t)}{W(y_1,y_2)(t)} \end{cases}$$

which by integrating we have

$$\begin{cases} u_1(t) = -\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt \\ u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt \end{cases}$$

□

- **Example 1:** (previous Exam Q.) Find a particular solution to

$$y'' + 4y = \frac{1}{\cos(2t)}.$$

  - **Step1:** First find y_c is possible. In this case y_c will be given by solving $r^2 + 4 = 0$ so that $r = \pm 2i$ hence

$$y_c(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

    Thus $y_1(t) = \cos(2t)$ and $y_2(t) = \sin(2t)$.
  - **Step2:** Find the Wronskian:

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} \\ &= 2\cos^2(2t) + 2\sin^2(2t) \\ &= 2\left[\cos^2(2t) + \sin^2(2t)\right] \\ &= 2 \cdot 1 = 2. \end{aligned}$$

  - **Step3:** Use our formula with $g(t) = \frac{1}{\cos(2t)}$ and get

$$\begin{aligned} y_p(t) &= -y_1(t)\left[\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt\right] + y_2(t)\left[\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt\right] \\ &= -\cos(2t)\left[\int \frac{1}{2}\frac{\sin(2t)}{\cos(2t)}dt\right] + \sin(2t)\left[\int \frac{\cos(2t)}{2}\frac{1}{\cos(2t)}dt\right] \\ &= -\cos(2t)\left[\frac{1}{2}\int \frac{\sin(2t)}{\cos(2t)}dt\right] + \frac{t}{2}\sin(2t) \end{aligned}$$

    now you can remenber the antiderivative of $\int \tan(2t)dt$ ot use u-substitution with $u = \cos(2t)$ and get $du = -2\sin(2t)dt$ so that

$$\int \frac{\sin(2t)}{\cos(2t)}dt = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln|\cos(2t)|$$

    hence

$$y_p(t) = \frac{1}{4}\cos(2t)\ln|\cos(2t)| + \frac{t}{2}\sin(2t).$$

- **Example 2:** (previous Exam Q.) Given that a general solution of the homogeneous equation, $t^2y'' + 2ty' - 2y = 0$ on $t > 0$ is

$$y_c = c_1 t + c_2 t^{-2}.$$

  Using Variation of Parameters, a particular solution $u_1 t + u_2 t^{-2}$ of the equation

$$t^2y'' + 2ty' - 2y = 6t$$

is found. What are the derivatives u_1' and u_2'?

- **Solution:** Notice that by variation of parameters we know that

$$\begin{aligned} y_p(t) &= -y_1(t)\left[\int \frac{y_2(t)g(t)}{W(y_1,y_2)(t)}dt\right] + y_2(t)\left[\int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt\right] \\ &= y_1(t)u_1(t) + y_2(t)u_2(t) \end{aligned}$$

  where

$$u_1(t) = \int \frac{-y_2(t)g(t)}{W(y_1,y_2)(t)}dt \text{ and } u_2(t) = \int \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}dt.$$

  hence

$$u_1'(t) = \frac{-y_2(t)g(t)}{W(y_1,y_2)(t)} \text{ and } u_2(t) = \frac{y_1(t)g(t)}{W(y_1,y_2)(t)}$$

  Thus to find the derivatives we simply need to use these formulas.

- Here's way they try to trick you here
  * First using that $y_1 = t$ and $y_2 = t^{-2}$
  * and $g(t) = 6t$ .Wrong!!!! We need to rewrite the equation as

$$y'' + \frac{2}{t}y' - 2y = \frac{6}{t}$$

    hence

$$g(t) = \frac{6}{t}.$$

- Compute the Wronskian:

$$W(y_1,y_2)(t) = \begin{vmatrix} t & t^{-2} \\ 1 & -2t^{-3} \end{vmatrix} = -2t^{-2} - t^{-2} = -3t^{-2} \neq 0 \text{ on } t > 0$$

- Thus

$$u_1(t) = -\int \frac{t^{-2}\cdot}{-3t^{-2}}\frac{6}{t}dt = \int \frac{2}{t}dt$$

  and

$$u_2(t) = \int \frac{t}{-3t^{-2}}\frac{6}{t}dt = \int -2t^2 dt$$

  hence

$$\begin{cases} u_1'(t) = \frac{2}{t} \\ u_2'(t) = -2t^2. \end{cases}$$

Sec. 3.7 - Mechanical and Electrical Vibrations

Recall Summary of Shortcuts:
So if you have

$$au'' + bu' + cu = 0$$

then find the roots of

$$ar^2 + br + c = 0.$$

The general solutions are the following:

roots:	The general Solution	Example
r_1, r_2 =real, distint	$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$(r+1)(r-1) = r^2 - 1 = 0$
$r = \lambda \pm i\mu$,imaginary	$y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$	$r^2 + 1 = 0$
$r = r_1$,real, repeasted	$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$.	$(r-2)^2 = 0$

- **Goal:** The equation

$$ay'' + by' + cy = 0$$

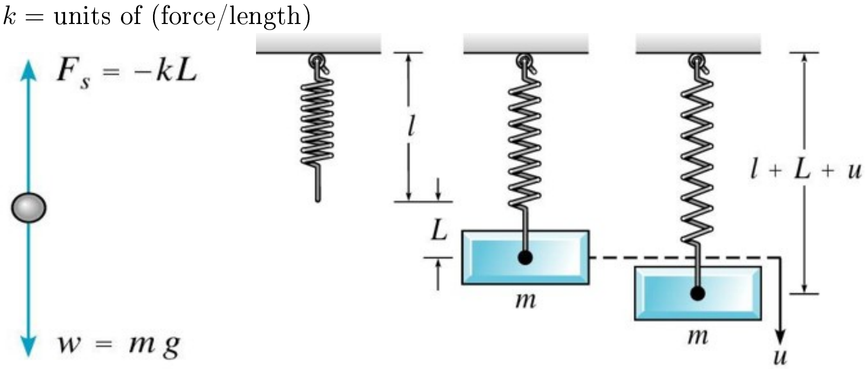
models a Harmonic oscillator: In particular, we will study the motion of a mass attached to a vibrating spring.

Spring-Mass System:

- Suppose a mass m hangs from a vertical spring of original length l. The mass causes an elongation L of the spring.
- The force F_G of gravity pulls the mass down. This force has magnitude mg, where g is acceleration due to gravity.
- The force F_s of the spring stiffness: always acts to pull spring to natural position. Force is upward.
  - For small elongations L, this force is proportional to L. That is, $F_s = -kL$ (Hooke's Law). Thus the mass is in equilibrium when the forces balance out:

$$mg = kL \qquad (\star)$$

  - We use this EQ to solve for k = units of (force/length)



- We have the follwing scenario:
- We will study the motion of a mass when it is acted on by an external force (forcing function) and/or is initially displaced.
  - $u(t)$ =displacement of the mass from its equilibrium position at time t, measure downward as being positive.
- Using Newton's second Law:

$$mu''(t) = f(t)$$

where f is the new force acting on the mass.

- The **forces** are:
    - Weight: $w = mg$ (downward force)
    - Spring force: $F_s = -k(L + u(t))$ (up or down force)
    - Damping force: $F_d(t) = -\gamma u'(t)$ (up or down): may be due to air resistance, friction, machanical device:
        - It acts in the opposite direction as the motion of the mass
    - External force: $F(t)$ (up or down)

- Putting it all together

$$\begin{aligned} mu''(t) &= mg + F_s(t) + F_d(t) + F(t) \\ &= mg - k(L + u((t)) - \gamma u'(t) + F(t) \end{aligned}$$

which using $mg = kL$ and simplifying we have

$$mu''(t) + \gamma u'(t) + ku(t) = F(t) \quad u(0) = u_0, u'(0) = v_0.$$

where m, γ, k are positive.
- m is found from $w = mg$
- γ is given in units of $\frac{\text{weight unit}\cdot s}{\text{distance unit}}$.
- k is found from $mg = kL$

- **Example1:** A 4 lb mass stretches a spring 2 inches. The mass is displaced an additional 6 in. and then released; and is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 3 ft/sec. Formulate the IVP that governs the motion of this mass:
    - **Solution:**
        - Find m: $w = mg$ which implies

$$m = \frac{w}{g} = \frac{4 \text{ lb}}{32 \text{ ft/s\^{}2}} = \frac{1}{8} \frac{\text{lb}s^2}{ft}$$

        - Find γ: Using $\gamma u' = 6\,\text{lb}$ we have

$$\gamma = \frac{6 \text{ lb}}{3 \text{ ft/sec}} = 2\frac{\text{lb sec}}{\text{ft}}.$$

        - Find k:

$$k = \frac{mg}{L} = \frac{4 \text{ lb}}{2\,\text{in}} = \frac{4 \text{ lb}}{(1/6) \text{ ft}} = 24\frac{\text{lb}}{ft}.$$

        - Thus

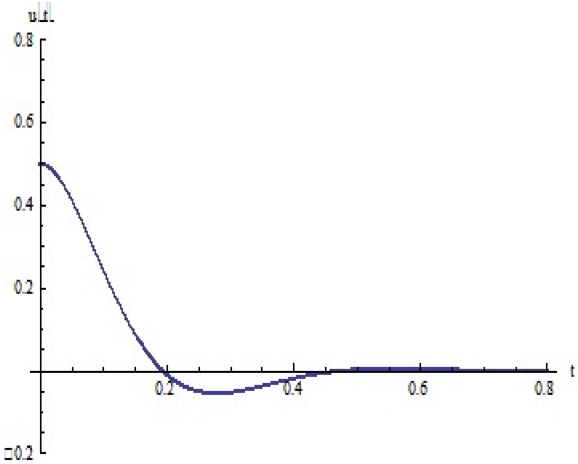
$$\frac{1}{8}u'' + 2u' + 24u = 0$$

hence

$$u'' + 16u' + 192u = 0, \quad u(0) = \frac{1}{2}, \quad u'(0) = 0$$

since $u(0) = 6\text{in}\ \frac{1\text{ft}}{12\text{in}} = \frac{1}{2}$.
        - Solving this

$$u(t) = \frac{1}{4}e^{-8t}\left(2\cos\left(8\sqrt{2}t\right) + \sqrt{2}\sin\left(8\sqrt{2}t\right)\right).$$



*

Undamped Mass-Spring:

When the damping coefficient $\gamma = 0$ (nothing stopping it from oscilaating forever) we have

$$mu'' + ku = 0$$

so that $mr^2 + k = 0$ gives u $r = \pm i\sqrt{\frac{k}{m}}$.This is a special number, so we'll denote it $\omega_0 = \sqrt{\frac{k}{m}}$. We get

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

with period $\frac{2\pi}{\omega}$.

- Phase plane of $(u(t), u'(t))$ look like ellipses.
- Mass either oscillates forever or stays at rest.
- Using trig:

$$u(t) == A\cos\omega_0 t + B\sin\omega_0 t = R\cos(\omega_0 t - \delta)$$

  where

$$A = \cos\delta, R = \sqrt{A^2 + B^2}$$
$$B = \sin\delta, \tan\delta = \frac{B}{A}.$$

- $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$. **The Period:**
- $\omega_0 =$ **natural frequency.**
- $R =$ **amplitude** of the maximum displacement of mass from equilibrium
- $\delta =$ **phase angle.** measures displacement of the wave from its normal position corresponding to $\delta = 0$. (note that the firs extrema happens at $t = \frac{\delta}{\omega_0}$)
- **Example2(popular exam question):** Determine ω_0, R and δ and rewrite as a cosine the expression

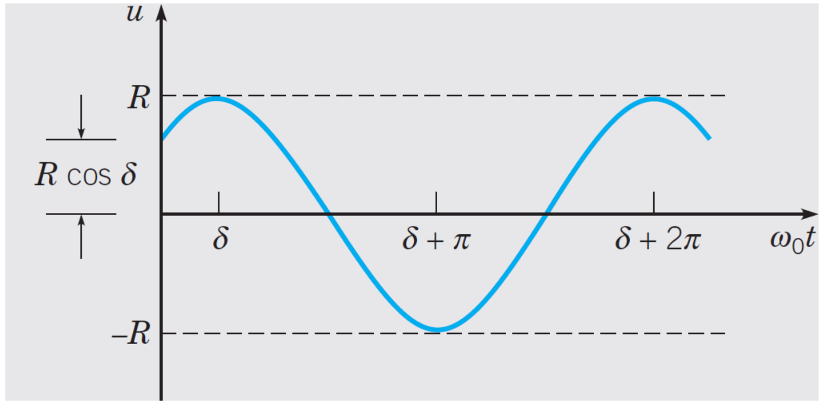
$$u(t) = 3\cos(2t) + 4\sin(2t).$$

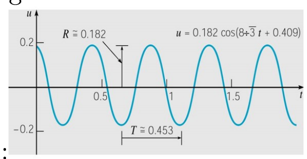
  - **Solution:** We have that

$$u(t) = R\cos(\omega_0 t - \delta)$$

    where $R = \sqrt{3^2 + 4^2} = 5$, $\omega_0 = 2$ and $\tan\delta = \frac{4}{3}$ hence $\delta = \tan^{-1}(\frac{4}{3})$ and we obtain

$$u(t) = 5\cos\left(2t - \tan^{-1}\left(\frac{4}{3}\right)\right).$$

– Something like this: 

– Example: 

Damped Harmonic oscillator:

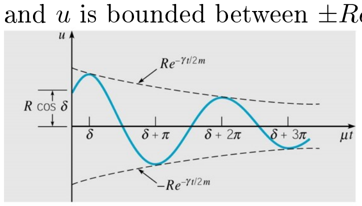
In general we'll have the following characeteristic equation

$$mr^2 + \gamma r + k = 0$$

solving for the roots we get

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$

Things change when the $b^2 - 4mk =, >, < 0$. We'll classify in the following way

- If $\gamma = 0$,
  - the oscillator is **undamped**.
  - The equilbrium point at the origin is a center (i.e. ellipse,circles). Possible **Graphs**:
  - Mass oscillates forever
  - The natural period is $2\pi\sqrt{\frac{m}{k}}$.
- If $\gamma > 0$ and $\gamma^2 - 4km < 0$ happens when there are $r = \alpha \pm \beta i$.
  - The oscillator is **underdamped**. The mass oscillates back and forth as it tends to its rest position.
  - Possible **Graphs**:
  - The most important case is $\gamma^2 - 4km < 0$:
    * Then
    $$u = Re^{-\gamma t/(2m)} \cos(\mu t - \delta)$$
    and u is bounded between $\pm Re^{-\gamma} Re^{-\gamma t/(2m)}$.
    * 

    * μ =called the **quasi-frequency:**

$$\mu = \frac{\sqrt{4km - \gamma^2}}{2m}$$

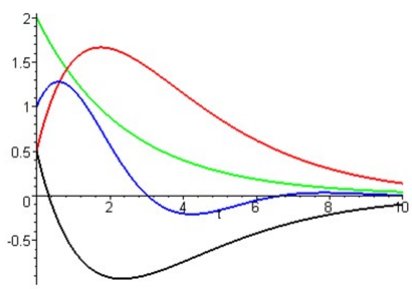
    * $T_d = (2\pi)/\mu$ is called the **quasi period.**
    * $\frac{T_d}{T} = \frac{\omega_0}{\mu}$

- If $\gamma > 0$ and $\gamma^2 - 4km > 0$ happens when there are two distinct r_1, r_2.
  - The oscillator is **overdamped**. The mass tends to its rest position but does not oscillate.
  - The equilbrium point is a real sink Possible **Graphs**:

$$u = Ae^{r_1 t} + Be^{r_2 t}, \;\; r_1, r_2 < 0$$

- If $\gamma > 0$ and $\gamma^2 - 4km = 0$ happens when there is one negative r.
  - The oscillator is **critically damped.** The mass tends to its rest position but does not oscillate.
  - Solutions tend to the origin tangent unique line of eigenvectors. **Graphs** looks like

$$u = Ae^{-\gamma t/(2m)} + Bte^{-\gamma t/(2m)}$$



- Graphs:
  - Underdamped: Blue
  - Overldamped: Green:
  - Critically damped: Red/Black
- **Example3:** ASuppose the motion of a spring-mass system is governed by

$$u'' + .125u' + u = 0, \;\; u(0) = 2, u'(0) = 0.$$

  - **Part(a):** Find the quasi frequency and quasi period.
    * We can get

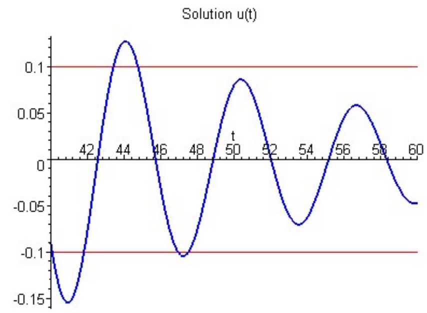
$$u(t) = e^{-t/6}\left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t\right)$$
$$= \frac{32}{\sqrt{255}}e^{-t/6}\left(\frac{\sqrt{255}}{16}t - \delta\right)$$

    now $\tan\delta = \frac{B}{A} = \frac{1}{\sqrt{255}}$ so that $\delta = .06254$

    * Quasi Frequency: $\mu = \frac{\sqrt{4km-\gamma^2}}{2m} = \sqrt{255}/16 = .998$
    * Quasi Period: $T_d = 2\pi/\mu = 6.295$

  - **Part(b):** Find time at which mass passes through equlibrium position

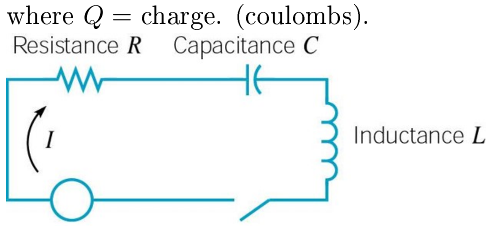
- * We set
$$\frac{32}{\sqrt{255}}e^{-t/6}\left(\frac{\sqrt{255}}{16}t-\delta\right)=0$$
and get
$$\frac{\sqrt{255}}{16}t-\delta=\frac{\pi}{2}$$
so that
$$t\approx 1.637 \text{ sec}$$
- **Part(c):** Find τ such that $|u(t)|<.1$ for all $t>\tau$



  * Use a computer/calculator:
  * Get $\tau = 47.515$.

Electric Circuts:

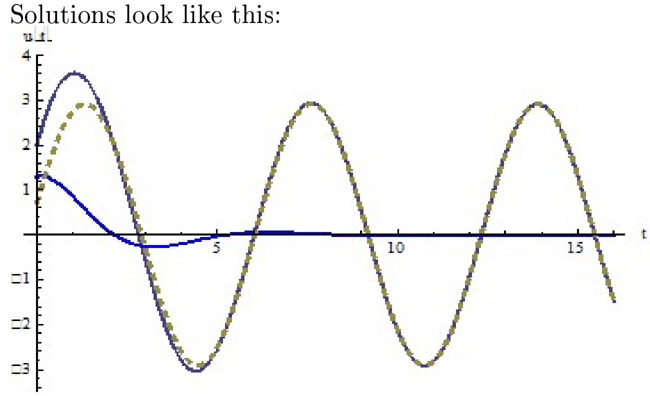
- The flow of **charge** in certain basic electrical circuits is modeled by second order linear ODEs with constant coefficients:
$$LQ''(t)+RQ'(t)+\frac{1}{C}Q(t)=E(t),\quad Q(0)=Q_0,\ Q'(0)=Q_0'$$
where Q = charge. (coulombs).



- Impressed voltage $E(t)$
- **Or**
- The flow of **current** in certain basic electrical circuits is modeled by second order linear ODEs with constant coefficients:
$$LI''(t)+RI'(t)+\frac{1}{C}I(t)=E'(t),\quad I(0)=I_0,\ I'(0)=I_0'$$
where $I(t)$ =current (amperes)

Sec. 3.8 - Forced Vibrations

- **Forced Vibrations with damping:**
  - We consider equations of the form
  $$mu'' + \gamma u' + ku = F(t)$$
  where $m > 0, \gamma > 0, k > 0$ are mass, damping coefficients, and spring constant. Here $F(t)$ represents external force done to the mass-spring system.
  - We consider the case when
  $$F(t) = F_0 \cos \omega t.$$
  - Recall that we can write the solution as
  $$\begin{aligned} u(t) &= c_1 u_1(t) + c_2 u_2(t) + A \cos \omega t + B \sin \omega t \\ &= u_c + u_p. \end{aligned}$$
  - To find, u_c: we solve $mr^2 + \gamma r + kr = 0$ and get
  $$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$
    - Since $m, \gamma, k > 0$ then
      - $r_1, r_2 =$, real, negative: $\lim_{t\to\infty} u_c(t) = \lim_{t\to\infty}(c_1 e^{r_1 t} + c_2 e^{r_2 t}) = 0$, or $\lim_{t\to\infty} u_c(t) = \lim_{t\to\infty}(c_1 e^{r_1 t} + c_2 t e^{r_2 t}) = 0$.
      - $r_1, r_2 =$ imaginary, negative real part: $\lim_{t\to\infty} u_c(t) = \lim_{t\to\infty}\left(c_1 e^{\lambda t}\cos \mu t + c_2 e^{\lambda t} \sin \mu t\right) = 0$.
      - In either case:
      $$\lim_{t\to\infty} u_c(t) = 0.$$
- **Summary:**
  - If solving: $mu'' + \gamma u' + ku = F_0 \cos \omega t$
  - **Solution:** $u(t) = u_c(t) + A\cos\omega t + B \sin \omega t$ where $\lim_{t\to\infty} u_c(t) = 0) = 0$.
    - We call $u_c(t)$ the **transient solution.**
    - $u_p(t) = A \cos \omega t + B \sin \omega t$ is called the **steady-state solution.**
      - This means that in the long term, the solution $u(t) \approx u_p(t)$ and $u_p(t)$ has the same frequency as the force F.
      - Without damping, the effect by intial conditions would persist for all time.
  - Solutions look like this:



  –
- **Studying Amplitude: (if time permits)**

- Using trig identities we have that the steady state solution can be rewritten

$$u_p(t) = A\cos(\omega t) + B\sin(\omega t)$$
$$= R\cos(\omega t - \delta)$$

where

$$R = \frac{F_0}{\sqrt{m^2\left(\omega_0^2 - \omega\right)^2 + \gamma^2\omega^2}} = \text{amplitude}$$

where

$$\cos\delta = \frac{m\left(\omega_0^2 - \omega^2\right)}{\sqrt{m^2\left(\omega_0^2 - \omega\right)^2 + \gamma^2\omega^2}} \text{ and } \sin\delta = \frac{\gamma\omega}{\sqrt{m^2\left(\omega_0^2 - \omega\right)^2 + \gamma^2\omega^2}}.$$

and $\omega_0^2 = \frac{k}{m}$.

- **Example1:**
  - A mass of 3 kg stretches a spring 5 cm.
  - The mass is acted on by external force of $7\cos(3t)$ N
  - It moves in a medium that imparts a viscous force of 3 N when the speed of the mass is 2 cm/s.
  - If the mass is set in motion from its equilibrium position w/ initial velocity 3 cm/s, formulate IVP:
  - **Solution:**
  - Recall 1 N $= \frac{1\ \text{kg}\cdot\text{m}}{s^2} \implies$ need to convert cm/s to m/s
  - $2\frac{cm}{s} = 2\frac{cm}{s}\frac{1m}{100cm} = .02\frac{\text{m}}{s}$.
  - $3\frac{cm}{s} = .03\frac{\text{m}}{s}$
  - $5\frac{cm}{s} = .05\frac{\text{m}}{s}$
  - Recall

$$mu'' + \gamma u' + ku = 7\cos(3t).$$

  - Remenber: $mg = kL \implies 3\cdot 9.8 = k\cdot .05$ so

$$k = \frac{3\cdot 9.8}{.05} = 588\frac{\text{kg}}{s^2}.$$

  - Damping Force= 3N when $|u'| = .02$, so

$$3 = \gamma\cdot(.02) \implies \gamma = 150\frac{N\cdot s}{m}.$$

  - Thus

$$3u'' + 150u' + 588u = 7\cos(3t).$$

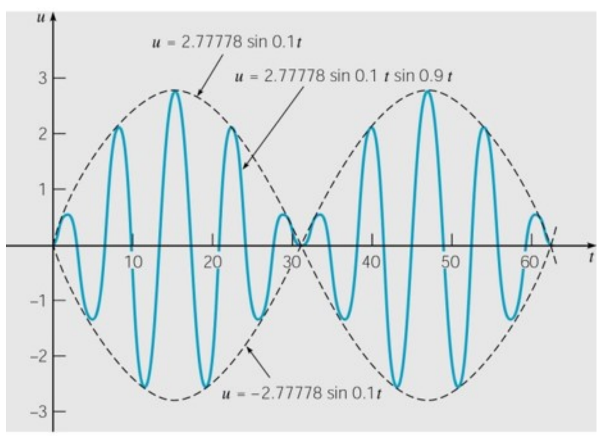
- **Forced Vibrations without damping:**
- **Example2:** Consider

$$u'' + 2u = \cos(\omega t), \quad \omega \neq \sqrt{2}.$$

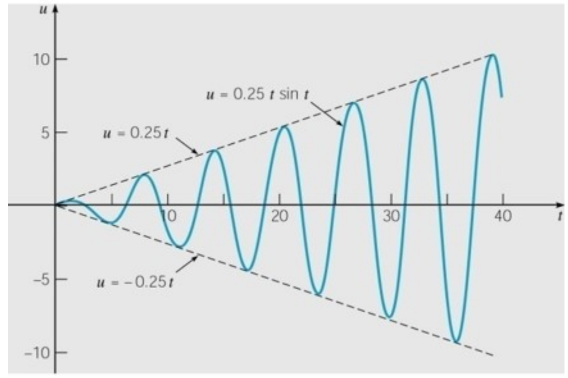
  - **Solution:**
  - **Step1:** Recall that $r^2 + 2 = 0$ so $r = \pm\sqrt{2}i$, so that

$$u_c(t) = c_1\cos\left(\sqrt{2}t\right) + c_2\sin\left(\sqrt{2}t\right).$$

- **Step2:** We make our first guess
$$u_p(t) = A\cos(\omega t) + B\sin(\omega t)$$
and there are no repeats with u_c as long as $\omega \neq 0$, hence we have the correct guess. Thus
$$u_p'(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$
$$u_p''(t) = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$$
hence plugging this into the LHS we have
$$\begin{aligned} LHS = u_p'' + 2u_p &= \left[-A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)\right] \\ &+ 2A\cos(\omega t) + 2B\sin(\omega t) \\ &= A\left(2-\omega^2\right)\cos(\omega t) + B\left(2-\omega^2\right)\sin(\omega t) \end{aligned}$$
setting $LHS = RHS = 1\cdot\cos(\omega t) + 0\sin(\omega t)$ we have
$$A\left(2-\omega^2\right) = 1, \quad B\left(2-\omega^2\right) = 0$$
$$A = \frac{1}{2-\omega^2}, \quad B = 0$$
so that
$$u(t) = c_1\cos\left(\sqrt{2}t\right) + c_2\sin\left(\sqrt{2}t\right) + \frac{1}{2-\omega^2}\cos(\omega t).$$

- **Possible Solutions:**
  - **Beating** happens when: natural frequency of spring system is approximately equal to frequency of the force:
$$\frac{\omega_0}{2\pi} \approx \frac{\omega}{2\pi} \iff \omega_0 \approx \omega.$$
  - The solution looks like this: 

    - ∗ This example had zero intiial conditions.
  - **Resonance:** happens when they are equal (this will be very interesting!):
$$\omega_0 = \omega.$$
  - But we need a new u_p to solve for when resonance happens, since can't plug $\omega = \sqrt{2}$.
$$\frac{1}{2-\omega^2}\cos(\omega t)$$



  - ∗ We will see that solution looks like this:
  - ∗ **Notice** that all of the physical phenomena we've observed so far either
    - · stayed oscillating forever (when undamped, not outside force),
    - · converged to zero (when damped, when no outside force happens),
    - · or the solution converges to a oscillating steady state solution (with outside force)
    - · Resonance, was the only one that blew up!
  - ∗ **Resonance** can be either good or bad, depending on circumstances; for example, when building bridges or designing seismographs.
    - · Go on youtube and search: Tacoma Narrows resonance.

- **Example3:** Consider

$$u'' + 2u = \cos\left(\sqrt{2}t\right), \quad u(0) = 0,\ u'(0) = 0.$$

  - **Solution:**
  - **Step1:** Recall that $r^2 + 2 = 0$ so $r = \pm\sqrt{2}i$, so that

$$u_c(t) = c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right).$$

  - **Step2:** We make our **first** guess

$$u_p(t) = A\cos\left(\sqrt{2}t\right) + B\sin\left(\sqrt{2}t\right)$$

  but we know there are repeats so we choose instead our **second** guess (by multiplying old guess by t)

$$u_p(t) = At\cos\left(\sqrt{2}t\right) + Bt\sin\left(\sqrt{2}t\right)$$

  - Thus

$$\begin{aligned} u_p'(t) &= A\cos\sqrt{2t} - A\sqrt{2}t\sin\sqrt{2}t \\ &\quad + B\sin\sqrt{2}t + B\sqrt{2}t\cos\sqrt{2}t \\ u_p''(t) &= -\sqrt{2}A\sin\sqrt{2}t - A\sqrt{2}\sin\sqrt{2}t \\ &\quad - A2t\cos\sqrt{2}t \\ &\quad B\sqrt{2}\cos\sqrt{2}t + B\sqrt{2}\cos\sqrt{2}t \\ &\quad - B2t\sin\sqrt{2}t \end{aligned}$$

hence plugging this into the LHS we have

$$\begin{aligned} LHS = u_p'' + 2u_p &= \text{simplify} \\ &= 2\sqrt{2}B\cos\sqrt{2}t - 2\sqrt{2}A\sin\sqrt{2}t \end{aligned}$$

setting $LHS = RHS = 1 \cdot \cos\left(\sqrt{2}t\right) + 0\sin\left(\sqrt{2}t\right)$ we have

$$2\sqrt{2}B = 1, \quad -2\sqrt{2}A = 0$$

$$B = \frac{1}{2\sqrt{2}}, \quad A = 0$$

so that

$$u(t) = c_1\cos\left(\sqrt{2}t\right) + c_2\sin\left(\sqrt{2}t\right) + \frac{1}{2\sqrt{2}}t\sin\left(\sqrt{2}t\right),$$

using initial consition we have $c_1 = 0, c_2 = 0$.

$$u(t) = \frac{1}{2\sqrt{2}}t\sin\left(\sqrt{2}t\right)$$

– Hence we get the picture similar to the one above since

$$\frac{1}{2\sqrt{2}}t\sin\left(\sqrt{2}t\right) \approx \pm\frac{t}{2\sqrt{2}} \text{ when } t \text{ is large.}$$

Sec. 4.1/4.2 - Higher Order Systems

- Everything did in Chapter 3 can be extended to higher order systems.
  - Suppose we have the equation
  $$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = g(t)$$
  - We assume that $a_n(t), \ldots, a_0(t)$ are continuous functions on an interval I, and that $a_n(t) \neq 0$ inside the interval: so that we can write it as
  $$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t). \quad (\star)$$
  with initial conditions
  $$y(t_0) = t_0,\ y'(t_0) = y_0',\ \cdots, y^{(n-1)}(t_0) = y_0^{(n-1)}. \quad (\star)$$
  - **Uniqueness/Existence Theorem:** If $p_{n-1}(t), \ldots, p_0(t)$ are continuous functions on an open interval I (containing t_0), then there exists a unique solution $y = \phi(t)$ throughout all of I to the IVP in $(\star)$.
- **Example1:** Consider to the ODE
$$(t-2)y^{(4)} + \sin t y''' + \ln t y = \sqrt{t+5}.$$
Find the intervals where you are guaranteed a unique solution to this ODE.
  - **Solution:** Rewriting we have
  $$y^{(4)} + \frac{\sin t}{(t-2)}y''' + \frac{\ln t}{(t-2)}y = \frac{\sqrt{t+5}}{(t-2)}$$
  and
    * $\frac{\sin t}{(t-2)}$ is comtinuous when $t \neq 2$
    * $\frac{\ln t}{(t-2)}$ is continuous when $t > 0$ and $t \neq 2$, and
    * $\frac{\sqrt{t+5}}{(t-2)}$ is continuous when $t \geq -5$ and $t \neq 2$.
    * Making a number line we see that all three functions are continuous when either on the interval $(0, 2)$ or $(2, \infty)$.
- Consider the **Homogeneous EQ with constant coefficients:**
$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$
  - As we did in the 2nd order case, the first thing we do is guess that the solution will look like $y = e^{rt}$ and
  $$\begin{aligned} y &= e^{rt} \\ y' &= re^{rt}, \\ &\vdots \\ y^{(n)} &= r^n e^{rt} \end{aligned}$$
  and plugging into the LHS and setting equal to zero we have
  $$\begin{aligned} LHS &= a_n\left(r^n e^{rt}\right) + \cdots + a_1\left(re^{rt}\right) + a_0\left(e^{rt}\right) \\ &= e^{rt}\left(a_n r^n + \cdots + a_0\right) \\ &= RHS = 0 \end{aligned}$$

hence
$$e^{rt}\left(a_n r^n + \cdots + a_0\right) = 0$$
but since $e^{rt} \neq 0$ then
$$a_n r^n + \cdots + a_1 r + a_0 = 0.$$

- As before the **characteristic equation** is given by:
$$\underbrace{a_n r^n + \cdots + a_0}_{Z(r)} = 0,$$
where we call $Z(t)$ the characteristic polynomial.
    * How do we solve $n-$degree polynomials? By factoring!
$$Z(t) = a_n\left(r - r_1\right)\left(r - r_2\right)\cdots\left(r - r_n\right).$$
    * **General Solution:** Solutions to the ODE are built exactly like in the 2nd degree case.
        · If there are any repeat solutions, then keep multiplying by t until you don't have any more repeat solutions.

- **Techniques for factoring:**
    1. (1) Factor by grouping
    2. (2) If you know $r = \alpha$ is a root, then we can divide the polynomial by $r - \alpha$ to get the other factor.
    3. (3) If $a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$
        (a) and $r = \frac{p}{q}$ is a rational root then possible roots are
$$r = \frac{\text{factors of } a_0}{\text{factors of } a_n}.$$
    4. (4) Can even result to using quadratic formula for things like $ar^4 + br^2 + c$ and get that
$$r^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
        (a) or use the substibution $x = r^2$ and factor $ax^2 + bx + c$.

- **Example1:** Find general solution and the particular solution to the IVP
$$2y''' - 4y'' - 2y' + 4y = 0. \quad y(0) = 0, y'(0) = 1, y'(0) = 2.$$
    - **Solution:** Characteristic equation is $2r^3 - 4r^2 - 2r + 4 = 0$ and note we can factor first divide by 2 to get
$$r^3 - 2r^2 - r + 2 = 0$$
and factor by grouping
$$\begin{aligned} r^2(r-2) - (r-2) &= (r-2)\left(r^2 - 1\right) \\ &= (r-2)(r+1)(r-1) = 0 \end{aligned}$$
hence the general solution is $y(t) = c_1 e^{2t} + c_2 e^{-t} + c_3 e^t$. To find the particular solution to the IVP we start by:
$$\begin{aligned} y(t) &= c_1 e^{2t} + c_2 e^{-t} + c_3 e^t \\ y'(t) &= 2c_1 e^{2t} - c_2 e^{-t} + c_3 e^t \\ y''(t) &= 4c_1 e^{2t} + c_2 e^{-t} + c_2 e^t. \end{aligned}$$

then we have to solve the following system of equations:

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 1 &= 2c_1 - c_2 + c_3 \\ 2 &= 4c_1 + c_2 + c_3 \end{aligned}$$

and get $c_1 = \frac{2}{3}$, $c_2 = -\frac{1}{6}$ and $c_3 = -\frac{1}{2}$, hence

$$y(t) = \frac{2}{3}e^{2t} - \frac{1}{6}e^{-t} - \frac{1}{2}e^{t}.$$

- **Example2:** Find general solution of

$$y^{(4)} + 8y''' + 16y'' = 0.$$

  - **Solution:** Characteristic polynomial

$$\begin{aligned} r^4 + 8r^3 + 16r^2 &= r^2\left(r^2 + 8r + 16\right) \\ &= r^2 (r+4)^2 = 0 \end{aligned}$$

  - Note that since this a 4th degree polynomial we need to have 4 roots: $0, 0, -4, -4$. So we use the same method we do when we have repeats and get

$$\begin{aligned} y(t) &= c_1 e^{0t} + c_2 t e^{0t} + c_3 e^{-4t} + c_4 t e^{-4t} \\ &= c_1 + c_2 t + c_3 e^{-4t} + c_4 t e^{-4t}. \end{aligned}$$

- **Example3:** Solve

$$y^{(4)} + y''' - 5y'' + y' - 6y = 0.$$

  - **Solution:** First we try to use our for rational roots on

$$Z(t) = r^4 + r^3 - 5r^2 + r - 6 = 0$$

which says possible roots are:

$$\frac{\pm 1}{\pm 1}, \frac{\pm 2}{\pm 1}, \frac{\pm 3}{\pm 1}, \frac{\pm 6}{\pm 1} = \pm 1, \pm 2, \pm 3, \pm 6.$$

By inspection we check that

$$r_1 = 2 \text{ and } r_2 = -3$$

are roots. Hence $Z(t)$ is divisible by $(r-2)(r+3) = r^2 + r - 6$ so we do Long Division:

$$\begin{array}{r} r^2+1 \\ r^2 + r - 6 \overline{\smash{)}\ r^4 + r^3 - 5r^2 + r - 6} \\ \underline{-\left(r^4 + r^3 - 6r^2\right)} \\ r^2 + r - 6 \\ \underline{-\left(r^2 + r - 6\right)} \\ 0 \end{array}$$

hence

$$Z(t) = (r-2)(r+3)\left(r^2+1\right)$$

which gives

$$r = 2, -3, \pm i$$

hence

$$y(t) = c_1e^{2t} + c_2e^{-3t} + c_3 \cos t + c_3 \sin t.$$

- **Example4:** Solve

$$y''' - 3y'' + 3y' - y = 0.$$

  - **Solution:** The characteristic polynomial is

$$r^3 - 3r^2 + 3r - 1$$

    * Note that by the rational root rule, we can check if $r = \frac{\pm 1}{\pm 1} = 1, -1$ are possible roots. By inspection only

$$r = 1$$

    is a root. We can then try to factor by grouping knowing we need a $(r-1)$. We can break up $Z(t)$ in the following way

$$\begin{aligned} r^3 \underbrace{-3r^2}_{\text{break this up}} + 3r - 1 &= \left(r^3 - r^2\right) - 2r^2 + 3r - 1 \\ &= r^2(r-1) - \left(2r^2 - 3r + 1\right) \\ &= r^2(r-1) - (2r-1)(r-1) \\ &= (r-1)\left[r^2 - (2r-1)\right] \\ &= (r-1)\left[r^2 - 2r + 1\right] \\ &= (r-1)(r-1)^2 \end{aligned}$$

  - So that

$$y(t) = c_1e^t + c_2te^t + c_3t^2e^t.$$

- **Example5:** (Exam like question) Solve

$$y^{(4)} + 8y'' - 9y = 0$$

  - **Solution:** Then use

$$r^4 + 8r^2 - 9 = 0$$

  then treat $u = r^2$ and since $u^2 + 8u - 9 = (u-1)(u+9)$ then

$$\begin{aligned} r^4 + 8r^2 - 9 &= \left(r^2 - 1\right)\left(r^2 + 9\right) \\ &= (r-1)(r+1)(r-3i)(r+3i). \end{aligned}$$

- **Example6**: Suppose the roots of the characteristic equation are

$$2, 3, 3, 3, 2 \pm 3i, 2 \pm 3i$$

then the general solution is

$$\begin{aligned} y(t) = {} & c_1e^{2t} + c_2e^{3t} + c_3te^{3t} + c_4t^2e^{3t} \\ & + c_5e^{2t}\cos(3t) + c_6e^{2t}\sin(3t) \\ & + c_5te^{2t}\cos(3t) + c_6te^{2t}\sin(3t). \end{aligned}$$

Sec. 4.3 - The Method of Undetermined Coefficients

- We consider
$$y^{(n)}+p_{n-1}(t)y^{(n-1)}+\cdots+p_1(t)y'+p_0(t)y=g(t)$$
where $g(t)$ can be a polynomial, sin, cos, exp or products of these.
  - Recall the General solution is of the form: $y=y_c+y_p$ where y_c is the general solution of the corresponding homogeneous equation and y_p is a particular solution to the non-homogeneous equation.
- **Example1:** Consider
$$y'''-y''-y'+y=2e^{-t}+3.$$
  - **Step1:** We find y_c: Solve
$$\begin{aligned} r^3-r^2-r+1 &= r^2(r-1)-(r-1) \\ &= (r-1)(r-1)(r+1)=0 \end{aligned}$$
so that $y_c=c_1e^t+c_2te^t+c_3e^{-t}$.
  - **Step2:** Find y_p: We **first guess** $y_p=Ae^{-t}+B$, but there are repeats with y_c hence we get a **second final guess** of
$$\begin{aligned} y_p &= Ate^{-t}+B \\ y_p' &= Ae^{-t}-Ate^{-t} \\ y_p'' &= -Ae^{-t}-Ae^{-t}+Ate^{-t}=-2Ae^{-t}+Ate^{-t} \\ y_p''' &= 2Ae^{-t}+Ae^{-t}-Ate^{-t}=3Ae^{-t}-Ate^{-t} \end{aligned}$$
Hence
$$\begin{aligned} LHS &= 3Ae^{-t}-Ate^{-t} \\ &\quad +2Ae^{-t}-Ate^{-t} \\ &\quad -Ae^{-t}+Ate^{-t} \\ &\quad +Ate^{-t}+B \\ &= 4Ae^{-t}+B \end{aligned}$$
  - **Step3:** Set LHS=RHS so that
$$LHS=4Ae^{-t}+B=2e^{-t}+3=RHS$$
hence
$$\begin{aligned} 4A=2&,B=3 \\ A&=\frac{1}{2} \end{aligned}$$
hence
$$y_p=\frac{1}{2}te^{-t}+3$$
so that the General Solution is
$$y=c_1e^t+c_2te^t+c_3e^{-t}+\frac{1}{2}te^{-t}+3.$$

- **Example2:** Consider
$$y''' + 4y' = t + \sin(4t).$$
Find the general form of y_p.
  - **Step1:** We find y_c: Solve
$$r^3 + 4r = 0$$
$$r\left(r^2+4\right) = 0$$
so that $y_c = c_1 + c_2\cos 2t + c_3 \sin 2t$.
  - **Step2:** Find y_p:
    * **First Guess:** $y_p = At + B + C\cos(4t) + D\sin(4t)$. But B is already in y_c as c_1.
    * **Second Guess:** $y_p = t\,(At+B) + C\cos(4t) + D\sin(4t)$ which is correct.
- **Example3:** Consider
$$y^{(4)} - 2y'' + y = e^t + te^{-t}.$$
Find the general form of y_p.
  - **Step1:** We find y_c: Solve
$$r^4 - 2r^2 + 1 = 0$$
$$\left(r^2-1\right)^2 = 0$$
so that $y_c = c_1e^t + c_2te^t + c_3e^{-t} + c_4te^{-t}$.
  - **Step2:** Find y_p:
    * **First Guess:** $y_p = Ae^t + (Bt + C)\,e^{-t}$.
    * **Second Guess:** $y_p = Ate^t + \left(Bt^2 + Ct\right)e^{-t}$.
    * **Third Guess:** $y_p = At^2e^t + \left(Bt^3 + Ct^2\right)e^{-t}$.
- **Example4:** Suppose we have the following characteristic equation:
$$Z(r) = r^3\left(r^2+1\right)^2(r+1)$$
and
$$g(t) = t^2 + 4 + e^t + \sin t + \cos 2t.$$
  - **Step1:** We find y_c: The roots are
$$r = 0, 0, 0, \pm i, \pm i, -1$$
so that
$$y_c = c_1 + c_2t + c_3t^2 + c_4\cos t + c_5 \sin t$$
$$+ c_6t\cos t + c_7t\sin t + c_8e^{-t}$$
  - **Step2:** Find y_p: I will **bold** whenever there are repeats with y_c.
    * **First Guess:** $y_p = \boldsymbol{At^2 + Bt + C} + De^t + \boldsymbol{E\cos t + F\sin t} + G\cos 2t + H\sin 2t$
    * **Second Guess:** $y_p = \boldsymbol{\left(At^2 + Bt + C\right)t} + De^t + \boldsymbol{t\,(E\cos t + F\sin t)} + G\cos 2t + H\sin 2t$
    * **Third Guess:** $y_p = \boldsymbol{\left(At^2 + Bt + C\right)t^2} + De^t + t^2\,(E\cos t + F\sin t) + G\cos 2t + H\sin 2t$.
    * **Fourth Guess:** $y_p = \left(At^2 + Bt + C\right)t^3 + De^t + t^2\,(E\cos t + F\sin t) + G\cos 2t + H\sin 2t$.
- **Example5:** Suppose
$$y^{(5)} = t^3$$

- **Step1:** We find y_c: The roots to $r^5 = 0$ are
$$r = 0, 0, 0, 0, 0$$
so that
$$y_c = c_1 + c_2 t + c_3 t^2 + c_4 t^3 + c_4 t^4$$
- **Step2:** Find y_p:
  * **First Guess:** $y_p = At^3 + Bt^2 + Ct + D$
  * **Final Guess:** $y_p = t^5 \left(At^3 + Bt^2 + Ct + D\right)$

Sec. 6.1 - Definition of the Laplace Transform

- We define $\mathcal{L}$, the Laplace transform.
- Before defining the Laplace Transform we review **improper integrals** since its definition depends on it.
- Improper Integrals:

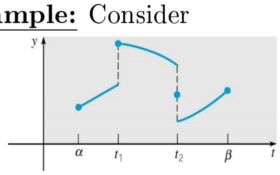
$$\int_a^\infty f(t)dt = \lim_{B\to\infty}\int_a^B f(t)dt.$$

  - If the limit converges then the improper integral converges.
  - If the limit diverges, then the improper integral diverges.
- **Definition:** f is **piecewise continuous** on $\alpha \le t \le \beta$ if it is continuous there except for a finite number of jump(or removable) discontinuities
- **Example:** Are the following functions piecewise continutions?

$$f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 1 & 1 < t \le 2 \\ 4-t & 2 < t \le 3 \end{cases}$$

  and

$$g(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ (t-1)^{-1} & 1 < t \le 2 \\ 1 & 2 < t \le 3. \end{cases}$$

  - **Solution:** Sketch the graphs
  - $f(t)$ is piecewise continuous since it only has a jump discontinuity.
  - $g(t)$ is **not** since it has a jump discontinuity.
- How do we integrate piecewise functions?
- **Example:** Consider
  - 

  - Then

$$\int_\alpha^\beta f(t)dt = \int_\alpha^{t_1} f(t)dt + \int_{t_1}^{t_2} f(t)dt + \int_{t_2}^{\beta} f(t)dt.$$

- A **transform** of $f(t)$ turns $f(t)$ into a different function.
  - We will transform functions$f(t)$ of t into functions $F(s)$ of s.
- **Definition:** The **Laplace transform** of f is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt.$$

- We assume s is real (though in general it can be complex).
- Thus the Laplace transforms a function $f(t)$ into a function $F(s)$

$$f(t) \xrightarrow{\mathcal{L}} F(s).$$

- **Goal:**

$$\begin{array}{ccc} \text{ODE Equation} & \overset{\mathcal{L}}{\Longrightarrow} & \text{Algebraic Equation} \\ & & \Downarrow \\ \text{Turn it into an ODE Solution} & \underset{\mathcal{L}^{-1}}{\Longleftarrow} & \text{Solve the Algebrac EQ} \end{array}$$

- **Existence of $\mathcal{L}\left\{f(t)\right\}$:**
  - If f is piecewise continuous for $[0,a]$ for all a.
  - $|f(t)| \leq Ke^{ct}$ for large t.
  - Then $\mathcal{L}\left[f(t)\right] = F(s)$ exists.
- **Example1:** Find the Laplace transform of $f(t) = e^{9t}$, $t \geq 0$.
  - **Solution:** We compute

$$\begin{aligned} \mathcal{L}\left\{e^{9t}\right\} &= \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{9t}e^{-st}dt \\ &= \int_0^\infty e^{(9-s)t}dt \\ &= \frac{1}{9-s}\left[e^{(9-s)t}\right]_{t=0}^{t=\infty} \\ &= \frac{1}{9-s}\left[\lim_{b\to\infty} e^{(9-s)b} - e^0\right] \end{aligned}$$

but since

$$\lim_{b\to\infty} e^{(9-s)b} = \begin{cases} \infty & a-s>0 \\ 0 & a-s<0 \end{cases}$$

then

$$\mathcal{L}\left\{e^{9t}\right\} = \begin{cases} \frac{1}{s-9} & s>9 \\ \text{not defined} & s<9 \end{cases}.$$

- **Example2:** Find the Laplace transform of $f(t) = e^{at}$, $t \geq 0$.
  - **Solution:** We can use the same computation as in Example 1, but change every 9 to an a and get

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \quad s>a.$$

- **Example3:** Find the Laplace transform of $f(t) = 1$, $t \geq 0$.
  - **Solution:** Using $a = 0$ above we have that $f(t) = e^{0\cdot t} = 1$ hence we can use the formula above to get

$$\mathcal{L}\left\{1\right\} = \frac{1}{s} \quad s>0.$$

- Eventually, we'll make a table where we collect all of the Laplace transforms that we have computed, so that we don't have to redo the work everytime.
- **Example4:** Find the Laplace transform of $f(t) = \sin(at)$.
  - **Solution:** We compute

$$\begin{aligned} \mathcal{L}\left\{\sin at\right\} = F(s) &= \int_0^\infty e^{-st}\sin(at)dt \\ &= \lim_{B\to\infty}\int_0^B e^{-st}\sin(at)dt \end{aligned}$$

hence using integration by parts

$$u = \sin(at) dv = e^{-st} dt$$
$$du = a\cos(at) dt v = -\frac{e^{-st}}{s}$$

we have

$$\begin{aligned} F(t) &= \lim_{B\to\infty}\left[-\frac{e^{-st}\sin(at)}{s}\Big|_{t=0}^{t=B} + \int_0^B \frac{e^{-st}}{s} a\cos(at)dt\right] \\ &= \lim_{B\to\infty}\left[-\frac{e^{-sB}\sin(aB)}{s} + 0 + \int_0^B \frac{e^{-st}}{s} a\cos(at)dt\right] \\ &= 0 + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)dt. \qquad (\star) \end{aligned}$$

– Integrating $\int_0^\infty e^{-st}\cos(at)dt$ again we get

$$u = \cos(at) dv = e^{-st} dt$$
$$du = -a\sin(at) dt v = -\frac{e^{-st}}{s}$$

$$\begin{aligned} \int_0^\infty e^{-st}\cos(at)dt &= \lim_{B\to\infty}\left[-\frac{e^{-st}\cos(at)}{s}\Big|_{t=0}^{t=B} - \int_0^B \frac{e^{-st}}{s} a\sin(at)dt\right] \\ &= \lim_{B\to\infty}\left[-\frac{e^{-sB}\cos(aB)}{s} + \frac{e^{-st}}{s} - \frac{a}{s}\int_0^B \frac{e^{-st}}{s} a\sin(at)dt\right] \\ &= \left[0 + \frac{1t}{s} - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)dt\right] \end{aligned}$$

hence plugging this back into $(\star)$ we have

$$\begin{aligned} F(t) &= \frac{a}{s}\left[\frac{1}{s} - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)dt\right] \\ &= \frac{a}{s}\left[\frac{1}{s} - \frac{a}{s}F(s)\right] \end{aligned}$$

hence we can solve this equation using algebra for $F(s)$ and get

$$F(s) = \frac{a}{s^2 + a^2}, \quad s > 0.$$

- **Properties of the Laplace Transform: Linearity**
  (1) If f, g are two function where $\mathcal{L}$ exists for $s > a_1$ and $s > a_2$, respectively, Then
  $$\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t))\} \pm \mathcal{L}\{g(t)\}, \quad s > \max\{a_1, a_2\},$$
  and
  (2) We have for $c \in \mathbb{R}$,
  $$\mathcal{L}\{cf(t))\} = c\mathcal{L}\{f(t))\}.$$
- **Example5:** Find the Laplace transform of $f(t) = 7 - e^{2t} + 4\sin(3t)$.

- **Solution:** Using what we have computed we get

$$\begin{aligned}\mathcal{L}\left\{7 - e^{-5t} + 4\sin(3t)\right\} &= \mathcal{L}\left\{7\right\} - \mathcal{L}\left\{e^{(-5)t}\right\} + 4\mathcal{L}\left\{\sin(3t)\right\} \\ &= \frac{7}{s} - \frac{1}{s-(-5)} + 4 \cdot \frac{3}{s^2+9} \\ &= \frac{7}{s} - \frac{1}{s+5} + \frac{12}{s^2+9}, \quad s > 0.\end{aligned}$$

Sec. 6.2 - Solutions to Initial Value Problems

- In this section we will show the connection between ODEs and Laplace Transforms.

Theorem. *1. Suppose f has a Laplace transform $\mathcal{L}\{f\}$ (see exact conditions in the textbook) then the Laplace transform of f' is given by*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Proof. Let

$$\mathcal{L}\{f'(t)\} = \lim_{B\to\infty}\int_0^B f'(t)e^{-st}dt$$

and we use integration by parts

$$u = e^{-st} \;\; dv = f'(t)dt$$
$$du = -se^{-st}dt \;\; v = f(t)$$

we have

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \lim_{B\to\infty}\left[f(t)e^{-st}\right]_{t=0}^{t=B} + \int_0^B f(t)se^{-st}dt \\ &= [0 - f(0)] + s\int_0^\infty f(t)e^{-st}dt \\ &= -f(0) + s\mathcal{L}\{f(t)\},\end{aligned}$$

here we use a condition from Theorem 6.2.1 that says $|f(t)| \le Ke^{at}$ for $t \ge M$ which implies that $\lim_{B\to\infty} f(B)e^{-sB} = 0$ when $s > a$. Rearranging gives us the desired result. □

Corollary. *Suppose $f, f', \ldots, f^{(n)}$ are nice functions that have Laplace transforms, then*

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0)\cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

- **Example:**
  - $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$.
  - $\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2f(0) - sf'(0) - f''(0)$.
- **Inverse Laplace Transforms:**
  - The Inverse Laplace transform $\mathcal{L}^{-1}$ is the function that satisfies $\mathcal{L}^{-1}\{\mathcal{L}[f]\} = f$. In other words,
  $$\mathcal{L}^{-1}\{F\} = f \iff \mathcal{L}\{f\} = F.$$
  - I like to think of $\mathcal{L}^{-1}$ of going backwords.
  - **Examples:**
    * $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$.
    * $\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$
    * $\mathcal{L}^{-1}\left\{\frac{10}{s+1}\right\} = 10\mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\} = 10e^{-t}$.
    * In general,
    $$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}.$$
- **Example1:** Solve
$$y' = y - 4e^{-t}, \;\; y(0) = 1$$
using Laplace transforms.

- **Solution:**
- **Step1:** Find the Laplace Transform of the ODE (The going forwards part):

$$\mathcal{L}\{y'\} = \mathcal{L}\{y\} - 4\mathcal{L}\{e^{-t}\} \iff s\mathcal{L}\{y\} - y(0) = \mathcal{L}\{y\} - 4\frac{1}{s+1}$$
$$\iff s\mathcal{L}\{y\} - 1 = \mathcal{L}\{y\} - 4\frac{1}{s+1}.$$

- **Step2:** Solve for $\mathcal{L}\{y\}$ using algebra: and get

$$\mathcal{L}\{y\} = \frac{1}{s-1} - \frac{4}{(s-1)(s+1)}.$$

- **Step3:** We want to go backwards and inverse this. But first let's do partial fractions:

$$\frac{4}{(s-1)(s+1)} = \frac{A}{(s-1)} + \frac{B}{(s+1)},$$

hence

$$4 = A(s+1) + B(s-1),$$
$$0 \cdot s + 4 = (A+B)s + (A-B)$$

so that

$$A + B = 0$$
$$A - B = 4$$

and get $A = 2, B - 2$. Thus

$$\frac{4}{(s-1)(s+1)} = \frac{2}{s-1} - \frac{2}{s+1}.$$

- **Step4:** Use the inverse Laplace transform to get

$$\begin{aligned} y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{4}{(s-1)(s+1)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \left(\mathcal{L}^{-1}\left\{\frac{2}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\}\right) \\ &= e^t - \mathcal{L}^{-1}\left\{\frac{2}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} \\ &= e^t - 2e^t + 2e^{-t} \\ &= -e^t + 2e^{-t}. \end{aligned}$$

- **Table:** We will use a table of Laplace transforms to perform more difficult Inverse Laplace Transforms. See **Page 321** from book.
- **More practice with taking Inverse Laplace Transform:**
  - **Step3:** We want to go backwards and inverse this.

(1) $F(s) = \frac{1}{s^4 + s^2}$ first let's do partial fractions:

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

hence

$$1 = As\left(s^2+1\right) + B\left(s^2+1\right) + (Cs+D)\,s^2$$

so that

$$0s^3 + 0s^2 + 0s + 1 = (A+C)\,s^3 + (B+D)\,s^2 + As + B$$

and get the equations

$$\begin{aligned} A + C &= 0 \\ B + D &= 0 \\ A &= 0 \\ B &= 1 \end{aligned}$$

and get $B = 1, A = 0, C = 0, D = -1$. Thus

$$\frac{1}{s^2\left(s^2+1\right)} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

Using **Formulas** 3 **and** 5 in the Laplace Trasform table:

$$\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}, \;\; \mathcal{L}\left\{\sin(at)\right\} = \frac{a}{s^2+a^2}.$$

Use get the inverse Laplace transform:

$$\begin{aligned} \mathcal{L}^{-1}\left\{F(s)\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= t - \sin t, \end{aligned}$$

(2) (Harder) $F(s) = \dfrac{1-2s}{s^2+4s+5}$. Note that we can't factor s^2+4s+5 with real roots, thus we will complete the square.
   - **Completing the Square:** Suppose we have s^2+bs+c, then the trick is to ADD/SUBTRACT $\left(\frac{b}{2}\right)^2$, and the polynomials will become $s^2+bs+c = \left(s+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$.
     * **Example:** Complete the square for s^2+4s+5: Then $b = 4$ hence we add/subtract $\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 4$. Thus

$$\begin{aligned} s^2 + 4s + 5 &= s^2 + 4s + 4 + (-4+5) \\ &= (s+2)^2 + 1 \end{aligned}$$

   - Going back to the problem of find the Laplace Transform we have that

$$F(s) = \frac{1-2s}{s^2+4s+5} = \frac{1-2s}{(s+2)^2+1}$$

   and looking at **Numbers 9**, and **Numbers 10** from the Laplace trasnform table:

$$\mathcal{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2} \text{ and } \mathcal{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2}.$$

We can apply these by separating $F(s)$ into pieces like this:=

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{1-2s}{(s+2)^2+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{-2(s+2)}{(s+2)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{\mathbf{+4}+1}{(s+2)^2+1}\right\} \\ &= -2\mathcal{L}^{-1}\left\{\frac{(s-(-2))}{(s-(-2))^2+1}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} \\ &= -2e^{-2t}\cos t + 5e^{-2t}\sin t.\end{aligned}$$

(3) $F(s) = \dfrac{2s-3}{s^2-4}$.
  - Notice that this one looks like **Numbers 7 and 8** from the Table of Laplace-Transforms:
  $$\mathcal{L}\{\sinh(at)\} = \frac{a}{s^2-a^2} \text{ and } \mathcal{L}\{\cosh(at)\} = \frac{s}{s^2-a^2} \quad s > |a|.$$
  - Hence we can separate $F(s)$ into pieces so that we can make it look like the formulas above:
  $$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-4}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2-2^2}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2-2^2}\right\} \\ &= 2\cosh(2t) - \frac{3}{2}\sinh(2t).\end{aligned}$$

(4) (if time permits)$F(s) = \dfrac{3s}{s^2-s-6}$. We want to use partial fractions
$$\frac{3s}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$
and multiply both sides by the denominator of the LHS we get
$$3s = A(s+2) + B(s-3)$$
and rewriting we get
$$3s + 0 = (A+B)s + (2A-3B)$$
so that
$$3 = A + B \text{ and } = 2A - 3B$$
and solving for A, B gets us
$$A = \frac{9}{5},\ \ B = \frac{6}{5}.$$
So that using our table we have that
$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{9/5}{s-3}\right\} + \mathcal{L}^{-1}\left\{\frac{6/5}{s-(-2)}\right\} \\ &= \frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}.\end{aligned}$$

- More Examples using Laplace Transforms to solve IVPs:

- **Example2:** Use Laplace Transforms to solve:

$$y''' + y' = 1, \;\; y(0) = y'(0) = y''(0) = 0.$$

  - **Solution:**
  - **Step1:** Find the Laplace Transform of the ODE (The going forwards part). Recall the formulas $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ and $\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$. Applying $\mathcal{L}$ to both sides we get

$$\begin{aligned}\mathcal{L}\{y''' + y'\} &= \mathcal{L}\{1\}, \iff \\ \left[s^3\mathcal{L}\{y\} - s^2 y(0) - sy'(0) - y''(0)\right] + \left[s\mathcal{L}\{y\} - y(0)\right] &= \frac{1}{s}, \iff \\ \left[s^3\mathcal{L}\{y\} - s^2 \cdot 0 - s \cdot 0 - 0\right] + \left[s\mathcal{L}\{y\} - 0\right] &= \frac{1}{s}, \iff \\ \mathcal{L}\{y\}\left(s^3 + s\right) &= \frac{1}{s}, \iff\end{aligned}$$

  - **Step2:** Solve for $\mathcal{L}\{y\}$ using algebra: and get

$$\mathcal{L}\{y\} = \frac{1}{s^2\left(s^2+1\right)}.$$

  - **Step3:** We want to go backwards and inverse this. But first let's do partial fractions: But remenber we did this in Example 1 of the Laplace transforms and got

$$\frac{1}{s^2\left(s^2+1\right)} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

  - **Step4:** Using **Formulas** 3 **and** 5 in the Laplace Trasform table:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \;\; \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}.$$

  Use get the inverse Laplace transform:

$$\begin{aligned}y = \mathcal{L}^{-1}\{\mathcal{L}\{y\}\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= t - \sin t,\end{aligned}$$

- **Example3:** Take the Laplace transform of the following equation:

$$y'' + y = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t < \infty \end{cases} \qquad y(0) = y'(0) = 0.$$

  - **Solution:** We first let the RHS be equal to

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & 1 \le t < \infty \end{cases}.$$

– We need to take $\mathcal{L}$ of both sides. We first take $\mathcal{L}$ of the RHS using the definition of the Laplace transform to obtain

$$\begin{aligned} \mathcal{L}\left\{f(t)\right\} &= \int_0^\infty f(t)e^{-st}dt \\ &= \int_0^1 te^{-st}dt + \int_1^\infty 0\cdot e^{-st}dt \\ &= \int_0^1 te^{-st}dt + 0, \text{ use integration by parts and get} \\ &= -\frac{(s+1)}{s^2}e^{-s} + \frac{1}{s^2}, \end{aligned}$$

– Thus we have that

$$\begin{aligned} y'' + y = f(t), &\iff \\ \mathcal{L}\left\{y''\right\} + \mathcal{L}\left\{y\right\} = \mathcal{L}\left\{ft)\right\}, &\iff \\ s^2\mathcal{L}\left\{y\right\} - sy(0) - y'(0) + \mathcal{L}\left\{y\right\} = -\frac{(s+1)}{s^2}e^{-s} + \frac{1}{s^2}, &\iff \\ \mathcal{L}\left\{y\right\}\left(s^2+1\right) = -\frac{(s+1)}{s^2}e^{-s} + \frac{1}{s^2}. \end{aligned}$$

and solving for $\mathcal{L}\left\{y\right\}$ we get

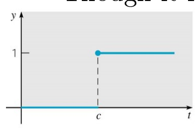
$$\mathcal{L}\left\{y\right\} = \frac{1-(s+1)e^{-s}}{s^2\left(s^2+1\right)}.$$

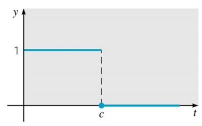
– We will learn how to invert this in the next section.

Sec. 6.3 - Step Functions

- Step functions are often used in problems involving
  - flow of electric circuits,
  - discontinuous impulsive forcing, such as in vibrations of mechanical systems
- **Definition:** The **Heaviside function**, or **unit step function** is defined by

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}.$$

  - Though it really doesn't matter, we will assume $c > 0$.
- 

- Note that $1 - u_c(t)$ looks like: 

- **Example1:** Sketch the following function and describe it as a piecewise function:

$$f(t) = 2tu_2(t) - (t-1)u_4(t).$$

  - **Solution:** We look at the critical points which are $t = 2, 4$ and consider different cases:
    * $t < 2$, $f(t) = 0 + 0 = 0$
    * $2 \leq t < 4$, $f(t) = 2t \cdot 1 + 0 = 2t$,
    * $4 \leq t$, $f(t) = 2t \cdot 1 - (t-1) \cdot 1 = t + 1$, hence

$$f(t) = \begin{cases} 0 & t < 2 \\ 2t & 2 \leq t < 4 \\ t+1 & t \geq 4. \end{cases}$$

- **Example2:** Write $f(t)$ in terms of step funtions:

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ t-1 & 1 \leq t < 2 \\ t-2 & 2 \leq t < 3 \\ 0 & 3 \leq t. \end{cases}$$

  - **Solution:** The critical points are $t = 0, 1, 2, 3$.
    * When $0 \leq t < 1$, the function will be $f(t) = tu_0(t) + \cdots$. Our goal is to figure out the rest.
    * When $1 \leq t < 2$, the function will be $f(t) = tu_0(t) + \mathbf{?} \cdot u_1(t) + \cdots = t - 1$, hence

$$t + ? = t - 1 \quad \Longrightarrow ? = -1.$$

      · Hence $f(t) = tu_0(t) - 1 \cdot u_1(t) + \cdots$
    * When $2 \leq t < 3$, the function will be $f(t) = tu_0(t) - 1 \cdot u_1(t) + \mathbf{?}u_2(t) + \cdots = t - 2$, hence

$$t - 1 + ? = t - 2 \quad \Longrightarrow ? = -1.$$

      · Hence $f(t) = tu_0(t) - 1 \cdot u_1(t) - 1u_2(t) + \cdots$

  * When $t \geq 3$, the function will be $f(t) = tu_0(t) - 1 \cdot u_1(t) - 1u_2(t) + \mathbf{?}u_3(t) \cdots = 0$, hence
$$t - 1 - 1 + ? = 0 \quad ? = 2 - t$$
  * Thus
$$f(t) = tu_0(t) - u_1(t) - u_2(t) + (2-t)u_3(t).$$

- We can compute the Laplace transform of $u_c(t)$:
$$\begin{aligned}\mathcal{L}\left\{u_c(t)\right\} &= \int_0^\infty u_c(t)e^{-st}dt = \int_c^\infty e^{-st}dt \\ &= \left[-\frac{e^{-st}}{s}\right]_{t=c}^{t=\infty} \\ &= \frac{e^{-cs}}{s}.\end{aligned}$$
- **Example3:** Find the Laplace Transform of
$$f(t) = \begin{cases} 2 & t < 3 \\ -3 & t \geq 3 \end{cases}.$$
  - **Solution:** First use the technique from the first two examples two write $f(t)$ in terms of u_c, and get
$$f(t) = 2 - 5u_3(t),$$
hence
$$F(s) = \mathcal{L}\left\{f(t)\right\} = \frac{2}{s} - 5\frac{e^{-3s}}{s}.$$

Theorem. *1. If $F(s) = \mathcal{L}\left\{f(t)\right\}$ exists for $s > a \geq 0$ and $c > 0$, then*
$$\mathcal{L}\left\{u_c(t)f(t-c)\right\} = e^{-cs}\mathcal{L}\left\{f(t)\right\} = e^{-cs}F(s),$$
Conversely, if $f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}$, then
$$\left\{u_c(t)f(t-c)\right\} = \mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\}.$$

- Remark: Note that $u_c(t)f(t-c)$ translates a function to the right by c, and leaves everything to the left as zero.

Theorem. *2. If $F(s) = \mathcal{L}\left\{f(t)\right\}$ exists for $s > a \geq 0$ and $c > 0$, then*
$$\mathcal{L}\left\{e^{ct}f(t)\right\} = F(s-s), \quad s > a + c.$$
Conversely, if $f(t) = \mathcal{L}^{-1}\left\{F(s-c)\right\}$, then
$$e^{ct}f(t) = \mathcal{L}^{-1}\left\{F(s-c)\right\}.$$

- We'll need the following **formulas:**
  - $\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$, n positive integer.
    * $\mathcal{L}\left\{t\right\} = \frac{1}{s^2}$, $\mathcal{L}\left\{t^2\right\} = \frac{2}{s^3}$, and $\mathcal{L}\left\{t^3\right\} = \frac{3!}{s^4}$.
  - $\mathcal{L}\left\{\sin at\right\} = \frac{a}{s^2+a^2}$
  - $\mathcal{L}\left\{\cos at\right\} = \frac{s}{s^2+a^2}$
  - $\mathcal{L}\left\{\sinh at\right\} = \frac{a}{s^2-a^2}$
  - $\mathcal{L}\left\{\cosh at\right\} = \frac{s}{s^2-a^2}$
  - $\mathcal{L}\left\{u_c(t)f(t-c)\right\} = e^{-cs}F(s)$

- $\mathcal{L}\left\{e^{ct} f(t)\right\} = F(s-c)$

- **Example4:** Find the Laplace transform of

$$f(t) = \begin{cases} 0 & t < 2 \\ t^2 - 4t + 5 & t \geq 2 \end{cases}$$

  - **Solution:** First we complete the square by adding/subtracting $\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 4$ and get

$$t^2 - 4t + 5 = t^2 - 4t + 4 - 4 + 5 = (t-2)^2 - 4 + 5 = (t-2)^2 + 1$$

so that

$$\begin{aligned} f(t) &= \begin{cases} 0 & t < 2 \\ (t-2)^2 + 1 & t \geq 2 \end{cases} \\ &= u_2(t)\left[(t-2)^2 + 1\right] \\ &= u_2(t)(t-2)^2 + u_2(t), \end{aligned}$$

hence using formulas $\mathcal{L}\left\{u_c(t) f(t-c)\right\} = e^{-cs} F(s)$ and $\mathcal{L}\left\{t^2\right\} = \frac{2}{s^3}$ and $c = 2$,

$$\begin{aligned} \mathcal{L}\left\{f(t)\right\} &= e^{-2s} F(s) + \frac{e^{-2s}}{s}, \text{ where } f(t-2) = (t-2)^2, f(t) = t^2 \\ &= e^{-2s} \frac{2}{s^3} + \frac{e^{-2s}}{s}. \end{aligned}$$

- **Example5:** Take the Inverse Laplace Transform of: $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$
  - **Solution:** In this example we can actually factor

$$\begin{aligned} \frac{e^{-2s}}{s^2 + s - 2} &= \frac{e^{-2s}}{(s+2)(s-1)} \\ &= e^{-2s}\left(\frac{-1/3}{s+2} + \frac{1/3}{s-1}\right), \text{ by partial fractions} \end{aligned}$$

and use $\mathcal{L}\left\{u_c(t) f(t-c)\right\} = e^{-cs} F(s)$ (use this whenever use see an e^{-cs} when taking **inverses**!)

$$\begin{aligned} \mathcal{L}^{-1}\left\{F(s)\right\} &= -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-(-2)}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}\right\} \\ &= -\frac{1}{3} u_2(t) f_1(t-2) + \frac{1}{3} u_2(t) f_2(t-2) \end{aligned}$$

  - Use the fact that $\mathcal{L}\left\{f_1\right\} = \mathcal{L}\left\{e^{-2t}\right\} = \frac{1}{s+2}$ and $\mathcal{L}\left\{f_2\right\} = \mathcal{L}\left\{e^{t}\right\} = \frac{1}{s-1}$ hence

$$\mathcal{L}^{-1}\left\{F(s)\right\} = -\frac{1}{3} u_2(t) e^{-2(t-2)} + \frac{1}{3} u_2(t) e^{(t-2)}.$$

- **Example6:** Take the Inverse Laplace Transform of: $F(s) = \dfrac{9(s-3)e^{-5s}}{s^2 - 6s + 13}$
  - **Solution:** In this example we can only complete the square since we can't factor and get

$$\frac{9(s-3)e^{-5s}}{s^2 - 6s + 13} = \frac{9(s-3)e^{-5s}}{(s-3)^2 + 2^2}$$

* Now note that by $\mathcal{L}\left\{e^{ct}f\left(t\right)\right\}=F(s-c)$ and $\mathcal{L}\left\{\cos at\right\}=\frac{s}{s^2+a^2}$ we have

$$\mathcal{L}\left\{\cos\left(2t\right)\right\}=\frac{s}{s^2+2^2}\implies\mathcal{L}\left\{e^{3t}\cos\left(2t\right)\right\}=\frac{(s-3)}{(s-3)^2+2^2}$$

* (Need to take care of the e^{-5s}) Now use $\mathcal{L}\left\{u_c(t)f_1\left(t-c\right)\right\}=e^{-cs}F_1(s)$ with $f_1(t)=e^{3t}\cos\left(2t\right)$ and $c=5$ so that $f_1(t-5)=e^{3(t-5)}\cos\left(2\left(t-5\right)\right)$ hence

$$\mathcal{L}\left\{u_5(t)e^{3(t-5)}\cos\left(2\left(t-5\right)\right)\right\}=e^{-5s}\frac{(s-3)}{(s-3)^2+2^2}$$

Thus multiplying both sides by 9

$$\mathcal{L}^{-1}\left\{F(s)\right\}=9u_5(t)e^{3(t-5)}\cos\left(2\left(t-5\right)\right).$$

- **Example7:** Take the Inverse Laplace Transform of: $F(s)=\dfrac{e^{-7s}}{s^2-4}$.
  - **Solution:** We note $\mathcal{L}\left\{\sinh 2t\right\}=\frac{2}{s^2-4}$ and use $\mathcal{L}\left\{u_c(t)f_1\left(t-c\right)\right\}=e^{-cs}F_1(s)$ where $f_1(t)=\sinh 2t\implies f_1(t-7)=\sinh 2\left(t-7\right)$ to get that

$$\begin{aligned}\mathcal{L}\left\{u_7(t)f_1\left(t-7\right)\right\}=e^{-7s}F_1(s),&\implies\mathcal{L}\left\{u_7(t)\sinh 2\left(t-7\right)\right\}=\frac{2e^{-7s}}{s^2-4}\\&\implies\frac{1}{2}\mathcal{L}\left\{u_7(t)\sinh 2\left(t-7\right)\right\}=\frac{e^{-7s}}{s^2-4}\end{aligned}$$

hence

$$\mathcal{L}^{-1}\left\{F(s)\right\}=\frac{1}{2}u_7(t)\sinh 2\left(t-7\right).$$

- **Example8:** Take the Inverse Laplace Transform of: $F(s)=\dfrac{1}{s^2}+\dfrac{e^{-6s}}{\left(s-2\right)^3}$
  - **Solution:** We know $\mathcal{L}\left\{t\right\}=\frac{1}{s^2}$, $\mathcal{L}\left\{t^2\right\}=\frac{2}{s^3}$ and $\mathcal{L}\left\{e^{ct}f\left(t\right)\right\}=F(s-c)$ and $\mathcal{L}\left\{u_c(t)f\left(t-c\right)\right\}=e^{-cs}F(s)$ hence

$$\begin{aligned}\mathcal{L}^{-1}\left\{F(s)\right\}&=t+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2e^{-6s}}{\left(s-2\right)^3}\right\}\\&=t+\frac{1}{2}u_6(t)e^{2(t-6)}(t-6)^2.\end{aligned}$$

- **Example9:** Take the Inverse Laplace Transform of: $F(s)=\dfrac{1}{s^2-10s+26}$
  - **Solution:** (practice with using $\mathcal{L}\left\{e^{ct}g\left(t\right)\right\}=G(s-c)$)First we complete the square and get

$$\frac{1}{s^2-10s+26}=\frac{1}{\left(s-5\right)^2+1}$$

and use $\mathcal{L}\left\{\sin at\right\}=\frac{a}{s^2+a^2}$ with $a=1$ so that $\mathcal{L}\left\{\sin t\right\}=\frac{1}{s^2+1}$

* Then use the fact that $\mathcal{L}\left\{e^{ct}g\left(t\right)\right\}=G(s-c)$ with $c=5$ and $g(t)=\sin t$, thus we know that

$$\mathcal{L}\left\{e^{5t}\sin t\right\}=\frac{1}{\left(s-5\right)^2+1}$$

hence

$$\mathcal{L}^{-1}\left\{F(s)\right\}=e^{5t}\sin t.$$

SEC. 6.4 - ODES WTIH DISCONTINUOUS FORCING FUNCTIONS

- We will now do some examples involving intial value problems:
- **Example1:** Solve using Laplace Transforms:

$$y'' + 4y = 3u_5(t)\sin(t-5), \quad y(0) = 1, y'(0) = 0.$$

  - **Solution:**
  - **Step1:** Take $\mathcal{L}$ of both sides and solve for $\mathcal{L}$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = 3\mathcal{L}\{u_5(t)\sin(t-5)\}$$

and recall $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$, hence $a = 4$, $f(t-5) = \sin(t-5)$ hence $f(t) = \sin t$ and $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$ hence

$$\begin{aligned} s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} &= 3\frac{e^{-5s}}{s^2+1}, \implies \\ \left(s^2+4\right)\mathcal{L}\{y\} - s &= 3\frac{e^{-5s}}{s^2+1}, \implies \\ \mathcal{L}\{y\} &= 3\frac{e^{-5s}}{(s^2+4)(s^2+1)} + \frac{s}{s^2+4} \end{aligned}$$

  - **Step2:** We do partial fractions on

$$\frac{3}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

hence

$$\begin{aligned} 3 &= (As+B)\left(s^2+1\right) + (Cs+D)\left(s^2+4\right), \implies \\ 0\cdot s^3 + 0\cdot s^2 + 0\cdot s + 3 &= (A+C)s^3 + (B+D)s^2 + (A+4C)s + (B+4D) \end{aligned}$$

hence

$$\begin{aligned} A + C &= 0 \\ B + D &= 0 \\ A + 4C &= 0 \\ B + 4D &= 3 \end{aligned}$$

and get

$$A = 0 \quad B = -1, \quad C = 0, \quad D = 1$$

hence

$$\frac{3}{(s^2+4)(s^2+1)} = \frac{-1}{s^2+4} + \frac{1}{s^2+1}$$

– **Step3:** Take the inverse Laplace transform: Using $\mathcal{L}\left[u_a(t)f(t-a)\right]=e^{-as}F(s)$, and $\mathcal{L}\left\{\sin(at)\right\}=\frac{a}{s^2+a^2}$ and $\mathcal{L}\left\{\cos(at)\right\}=\frac{a}{s^2+a^2}$ we have

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\frac{-e^{-5s}}{s^2+4}+\frac{e^{-5s}}{s^2+1}+\frac{s}{s^2+4}\right\} \\ &= -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2e^{-5s}}{s^2+2^2}\right\}+\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} \\ &= -\frac{1}{2}u_5(t)\sin\left(2\left(t-5\right)\right)+u_5(t)\sin\left(t-5\right)+\cos\left(2t\right) \end{aligned}$$

- **Example2:** Solve using Laplace Transforms:

$$y^{(4)}-y=u_1(t)-u_2(t), \quad y(0)=0, y'(0)=0, y''(0)=0., y'''(0)=0..$$

– **Solution:**
– **Step1:** Take $\mathcal{L}$ of both sides and solve for $\mathcal{L}$

$$\mathcal{L}\left\{y^{(4)}\right\}-\mathcal{L}\left\{y\right\}=\mathcal{L}\left\{u_1(t)-u_2(t)\right\}$$

hence

$$\begin{aligned} s^4\mathcal{L}\left\{y\right\}-s^3y(0)-s^2y'(0)-sy''(0)-sy'''-y'(0)-\mathcal{L}\left\{y\right\} &= \frac{e^{-s}}{s}-\frac{e^{-2s}}{s} \implies \\ \left(s^4-1\right)\mathcal{L}\left\{y\right\} &= \frac{e^{-s}}{s}-\frac{e^{-2s}}{s}, \implies \\ \mathcal{L}\left\{y\right\} &= \frac{e^{-s}}{s\left(s^4-1\right)}-\frac{e^{-2s}}{s\left(s^4-1\right)} \end{aligned}$$

– **Step2:** We do partial fractions on

$$\frac{1}{s\left(s^4-1\right)}=\frac{1}{s\left(s^2-1\right)\left(s^2+1\right)}=\frac{1}{s\left(s+1\right)\left(s-1\right)\left(s^2+1\right)}=$$

hence

$$\frac{1}{s\left(s^4-1\right)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s-1}+\frac{Ds+E}{s^2+1}$$

after doing the work to get the partial fractions you get

$$\frac{1}{s\left(s^4-1\right)}=-\frac{1}{s}+\frac{1}{4}\frac{1}{s+1}+\frac{1}{4}\frac{1}{s-1}+\frac{1}{2}\frac{s}{s^2+1}$$

putting it back we need to take the inverse of

$$e^{-s}\left[-\frac{1}{s}+\frac{1}{4}\frac{1}{s+1}+\frac{1}{4}\frac{1}{s-1}+\frac{1}{2}\frac{s}{s^2+1}\right]-e^{-2s}\left[-\frac{1}{s}+\frac{1}{4}\frac{1}{s+1}+\frac{1}{4}\frac{1}{s-1}+\frac{1}{2}\frac{s}{s^2+1}\right]$$

- **Step3:** Take the inverse Laplace transform: Using $\mathcal{L}\left[u_a(t)f(t-a)\right] = e^{-as}F(s)$, and $\mathcal{L}\left\{\cos(at)\right\} = \frac{a}{s^2+a^2}$ and $\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$ we have

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{e^{-s}\left[-\frac{1}{s} + \frac{1}{4}\frac{1}{s+1} + \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{s}{s^2+1}\right]\right\} \\ &- \mathcal{L}^{-1}\left\{e^{-2s}\left[-\frac{1}{s} + \frac{1}{4}\frac{1}{s+1} + \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{s}{s^2+1}\right]\right\} \\ &= -u_1(t) + u_1(t)\left[\frac{1}{4}e^{-1(t-1)} + \frac{1}{4}e^{1(t-1)} + \frac{1}{2}\cos(t-1)\right] \\ &+ u_2(t) - u_2(t)\left\{\frac{1}{4}e^{-1(t-2)} + \frac{1}{4}e^{1(t-2)} + \frac{1}{2}\cos(t-2)\right\} \end{aligned}$$

- **Example3:** Find the Laplace transform of

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 3t & 1 \le t < \infty \end{cases}.$$

- **Solution:**
  * **Step1:** First let us rewrite this in terms of unit step functions.
    · When $0 \le t < 1$: the function is $f(t) = t$
    · When $1 \le t < \infty$: then function is $f(t) = t + ? \cdot u_1(t) = 3t$ hence $? = 2t$ so that

$$f(t) = t + 2tu_1(t).$$

  * **Step2:** Before we can take a Laplace transform, we notice that our fomula involves $\mathcal{L}\left\{u_c(t)g(t-c)\right\} = e^{-ct}\mathcal{L}\left\{g(t)\right\}$. Thus we will need to turn $2tu_1(t)$ into this form:

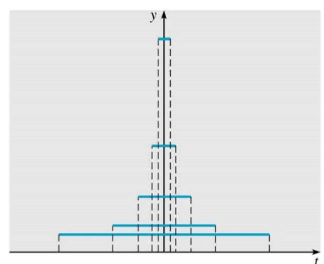
$$\begin{aligned} f(t) &= t + 2tu_1(t) \\ &= t + 2\left(t - \mathbf{1}\right)u_1(t) + \boldsymbol{2u_1(t)} \end{aligned}$$

hence

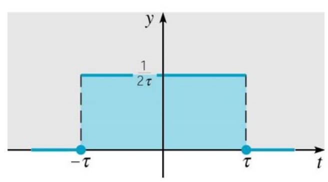
$$\begin{aligned} \mathcal{L}\left\{f(t)\right\} &= \mathcal{L}\left\{t\right\} + 2\mathcal{L}\left\{(t-1)u_1(t)\right\} + 2\mathcal{L}\left\{u_1(t)\right\} \\ &= \frac{1}{s^2} + 2e^{-s}\frac{1}{s^2} + 2e^{-s}\frac{1}{s}. \end{aligned}$$

Sec. 6.5 - Impulse Functions

- We consider
$$ay'' + by' + cy = f(t),$$
where
$$g(t) = \begin{cases} \text{large} & t_0 - \tau < t < t_0 + \tau. \\ 0 & \text{elsewhere} \end{cases}$$
- Here $g(t)$ is a **force** and
$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)dt = \int_{-\infty}^{\infty} g(t)dt$$
is the **impulse** of the force, or the amount of force in a short time period about t_0.
  - **Example:** If y =current in an electric circuit, $g(t) =$ is the time derivative of the voltage, then $I(\tau)$ is the total voltage impressed on circuit in the time interval $I = (t_0 - \tau, t_0 + \tau)$.
- We will use the following particular example of a force with $\tau = 0$ (to simplify things):
$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau, \\ 0 & \text{elsewhere} \end{cases},$$
where $\tau > 0$ is small.
  - **Nice Properties of $d_\tau(t)$:**



(1) $\lim_{\tau \to 0^+} d_\tau(t) = 0$, whenever $t \neq 0$, and $\lim_{\tau \to 0^+} d_\tau(0) = \infty$.
(2) $I(\tau) = \int_{-\tau}^{\tau} \frac{1}{2\tau} dt = \left[\frac{1}{2\tau} t\right]_{-\tau}^{\tau} = 1$ for every τ,



(a) Hence $\lim_{\tau \to 0^+} I(\tau) = 1$,
- We thus want to define a **unit impulse function** δ, with the properties
$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$
and
$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$

- This function doesn't really exist, it is an example of a *generalized* function called the **Dirac delta function**. But it can be "defined" as a limit of the $d_\tau(t)$ functions:

$$\delta(t) = \lim_{\tau\to 0^+} d_\tau(t).$$

- **In general:** We can consider a unit impulse at an arbitrary point $t = t_0$, meaning $\delta(t - t_0)$, hence

$$\delta(t - t_0) = 0, \ t \neq t_0,$$
$$\int_{-\infty}^{\infty} \delta(t - t_0)dt = 1.$$

- Let's computer the Laplace Trasnform of $\delta(t - t_0)$:

$$\begin{aligned}
\mathcal{L}\{\delta(t - t_0)\} &= \lim_{\tau\to 0^+} \mathcal{L}\{d_\tau(t - t_0)\} \\
&= \lim_{\tau\to 0^+} \int_{t_0-\tau}^{t_0+\tau} e^{-st} d_\tau(t - t_0)\, dt \\
&= \lim_{\tau\to 0^+} \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} e^{-st} dt = \lim_{\tau\to 0^+} \frac{1}{2\tau} \left[\frac{e^{-st}}{t}\right]_{t=t_0-\tau}^{t=t_0+\tau} \\
&= \frac{e^{-st_0}}{s} \lim_{\tau\to 0^+} \frac{e^{s\tau} - e^{-s\tau}}{2\tau}, \text{ by algebra} \\
&= e^{-st_0} \lim_{\tau\to 0^+} \frac{\sinh(s\tau)}{s\tau}, \text{ by fomrula below} \\
&= e^{-st_0} \lim_{\tau\to 0^+} \frac{s\cosh(s\tau)}{s}, \text{ by L'Hopitals rule} \\
&= e^{-st_0}.
\end{aligned}$$

where I used the fact that

$$\sinh(s\tau) = \frac{e^{s\tau} - s^{-s\tau}}{2}.$$

- **Summary:**

$$\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}, \ t_0 > 0.$$

- **Examples:**
  - If $t_0 = 0$, then

$$\mathcal{L}\{\delta(t)\} = e^{-s\cdot 0} = 1.$$

  - If $t_0 = 9$ then

$$\mathcal{L}\{\delta(t - 9)\} = e^{-9s}.$$

- **Important property of delta functions:** Suppose f is a continuous function, then

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0).$$

- In the next example we show how the delta function is connected to the Heaviside function.
- **Example0:** Solve the IVP

$$y' = \delta(t - c), \quad y(0) = y_0.$$

– **Solution:** Take $\mathcal{L}$ of both sides and

$$\mathcal{L}\{y'\} = \mathcal{L}\{\delta(t-c)\} \implies$$
$$s\mathcal{L}\{y\} - y(0) = e^{-cs} \implies$$
$$\mathcal{L}\{y\} = \frac{y_0}{s} + \frac{e^{-cs}}{s}$$

hence

$$y = \mathcal{L}^{-1}\left\{\frac{y_0}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-cs}}{s}\right\}$$
$$= y_0 + u_c(t).$$

- Example 0 shows that the derivative of the Heaviside function ins the delta function.
- **Fact:**

$$\frac{d}{dt}[u_c(t)] = \delta(t-c).$$

- **Example1:** Solve the IVP

$$y'' + 4y = \delta(t-\pi) - \delta(t-2\pi), \quad y(0) = 0, y'(0) = 0.$$

– **Solution:**

– **Step1:** Take $\mathcal{L}$ of both sides and solve for $\mathcal{L}\{y\}$:

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\pi)\} - \mathcal{L}\{\delta(t-2\pi)\} \implies$$
$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} = e^{-\pi s} - e^{-2\pi s} \implies$$
$$\left(s^2+4\right)\mathcal{L}\{y\} = e^{-\pi s} - e^{-2\pi s} \implies$$
$$\mathcal{L}\{y\} = \frac{e^{-\pi s}}{s^2+4} - \frac{e^{-2\pi s}}{s^2+4}.$$

– **Step2:** Notice that we don't need to do partial fractions or complete the square here since s^2+4 is already a sum of two squares.

– **Step3:** Take an inverse Laplace transform:

∗ Using $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$ and $\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$ we get

$$y = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+4}\right\}$$
$$= \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-\pi s}\frac{2}{s^2+2^2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{2}{s^2+2^2}\right\}$$
$$= \frac{1}{2}u_\pi(t)f_1(t-\pi) - \frac{1}{2}u_{2\pi}(t)f_2(t-2\pi)$$
$$= \frac{1}{2}u_\pi(t)\sin(2(t-\pi)) - \frac{1}{2}u_{2\pi}(t)\sin(2(t-2\pi))$$

where $f_1, f_2 = \sin(2t)$.

∗ Now it turns out, that

$$\sin(2(t-\pi)) = \sin(2t-2\pi) = \sin(2t)$$

and

$$\sin(2(t-2\pi)) = \sin(2t-4\pi) = \sin(2t).$$

or in general

$$\sin(x) = \sin(x + 2\pi).$$

* Hence a possible multiple choice answer could be:

$$y = \frac{1}{2}u_{\pi}(t)\sin(2t) - \frac{1}{2}u_{2\pi}(t)\sin(2t).$$

- **Example2:** Solve the IVP

$$y'' + 2y' + 3y = \sin t + \delta(t - 3\pi), \quad y(0) = 0, y'(0) = 0.$$

  - **Solution:**
  - **Step1:** Take $\mathcal{L}$ of both sides and solve for $\mathcal{L}\{y\}$:

$$\begin{aligned} \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} &= \frac{1}{s^2+1} + e^{-3\pi s} \implies \\ \left[s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right] + 2\left[s\mathcal{L}\{y\} - y(0)\right] + 3\mathcal{L}\{y\} &= \frac{1}{s^2+1} + e^{-3\pi s} \implies \\ s^2\mathcal{L}\{y\} + 2s\mathcal{L}\{y\} + 3\mathcal{L}\{y\} &= \frac{1}{s^2+1} + e^{-3\pi s} \implies \\ \left(s^2 + 2s + 3\right)\mathcal{L}\{y\} &= \frac{1}{s^2+1} + e^{-3\pi s} \implies \\ \mathcal{L}\{y\} = \frac{1}{(s^2+2s+3)(s^2+1)} &+ \frac{e^{-3\pi s}}{s^2+2s+3} \end{aligned}$$

  - **Step2:** First we do partial fractions:

$$\frac{1}{(s^2+2s+3)(s^2+1)} = \frac{As+B}{(s^2+2s+3)} + \frac{Cs+D}{(s^2+1)}$$

  and do the algebra to get

$$A = \frac{1}{4}, B = \frac{1}{4}, C = -\frac{1}{4}, D = \frac{1}{4}$$

  - Also we need to complete the square:

$$s^2 + 2s + 3 = (s+1)^2 + 2.$$

  so that

$$\frac{1}{(s^2+2s+3)(s^2+1)} = \frac{1}{4}\left(\frac{s+1}{(s+1)^2+2} + \frac{-s+1}{(s^2+1)}\right)$$

  - **Step3:** Take an inverse Laplace transform:
    * Using $\mathcal{L}[u_a(t)f(t-a)] = e^{-as}F(s)$, $\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$ $\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$,

* Also using $\mathcal{L}\left\{e^{at}\cos(bt)\right\} = \frac{s-a}{(s-a)^2+b^2}$ $\mathcal{L}\left\{e^{at}\sin(bt)\right\} = \frac{b}{(s-a)^2+b^2}$ we get

$$
\begin{aligned}
y &= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{-s+1}{(s^2+1)}\right\} \\
&+ \mathcal{L}^{-1}\left\{\frac{e^{-3\pi s}}{(s+1)^2+2}\right\} \\
&= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+\left(\sqrt{2}\right)^2}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} \\
&+ \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{e^{-3\pi s}\frac{\sqrt{2}}{(s+1)^2+\left(\sqrt{2}\right)^2}\right\} \\
&= \frac{1}{4}\left(e^{-t}\cos\left(\sqrt{2}t\right) - \cos t + \sin t\right) \\
&+ \frac{1}{\sqrt{2}}u_{3\pi}(t)\, f_1(t-3\pi) \\
&= \frac{1}{4}\left(e^{-t}\cos\left(\sqrt{2}t\right) - \cos t + \sin t\right) \\
&+ \frac{1}{\sqrt{2}}u_{3\pi}(t)\, e^{-1(t-3\pi)}\sin\left(\sqrt{2}(t-3\pi)\right)
\end{aligned}
$$

* Where $f_1 = \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+\left(\sqrt{2}\right)^2}\right\} = e^{-t}\cos\sqrt{2}t$.

Sec. 6.6 - The Convolution Integral

- Suppose we want to take the inverse Laplace transform of a product: Is it true that
$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\} \stackrel{?}{=} \mathcal{L}^{-1}\left\{F(s)\right\}\mathcal{L}^{-1}\left\{G(s)\right\}. \quad \text{,NO!}$$
- In order to take the inverse of a product, we need to define the **convolution integral**: Let $f(t), g(t)$ be two nice functions, then
$$(f \star g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau = \int_0^t f(\tau) g(t-\tau)\, d\tau.$$
  - The function $h = f \star g$ is called the **convolution** of f and g.
- **Theorem:** The Laplace transform of the convolution is
$$\mathcal{L}\left\{(f \star g)(t)\right\} = \mathcal{L}\left\{f(t)\right\}\mathcal{L}\left\{g(t)\right\} = F(s)G(s)$$
that is
$$\mathcal{L}^{-1}\left\{F(s)G(s)\right\} = (f \star g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau.$$
- Convolutions have nice properties: We can treat $\star$ almost like real multiplication
  - $f \star g = g \star f$ (commutative)
  - $f \star (g_1 + g_2) = f \star g_1 + f \star g_2$ (distributive)
  - $(f \star g) \star h = f \star (g \star h)$ (associative)
  - $f \star 0 = 0 \star f = 0$.
- However it doesn't have all the properties of ordinary multiplication: $(f \star 1) \neq 1 \star f$.
  - Consider $f = \cos t$.
- **Example1:** Find the Laplace transform of
$$h(t) = \int_0^t \sin(t-\tau)\cos\tau d\tau$$
  - **Solution:** Use $f = \sin t$ and $g = \cos t$ and we know that by the Theorem
$$\begin{aligned}\mathcal{L}\left\{\int_0^t \sin(t-\tau)\cos\tau d\tau\right\} &= \mathcal{L}\left\{\sin t\right\}\mathcal{L}\left\{\cos t\right\} \\ &= \frac{1}{s^2+1}\cdot\frac{s}{s^2+1} \\ &= \frac{s}{(s^2+1)^2}.\end{aligned}$$
- **Example2:** Find the Laplace transform of
$$e^t \int_0^t \sin\tau\cos(t-\tau)\, d\tau.$$
  - **Solution:** This question is testing if you know how to use formulas
$$\mathcal{L}\left\{e^{ct} f(t)\right\} = F(s-c)$$
  hence we need to first take the Laplace transform of
$$\mathcal{L}\left\{\int_0^t \sin\tau\cos(t-\tau)\, d\tau\right\} = \mathcal{L}\left\{\int_0^t \sin(t-\tau)\cos(\tau)\, d\tau\right\} = \frac{s}{(s^2+1)^2}$$

from Example1. Hence using the formula above we have

$$\mathcal{L}\left\{e^t \int_0^t \sin\tau\cos(t-\tau)\,d\tau\right\} = \frac{s-1}{\left((s-1)^2+1\right)^2}$$

- **Example3:** Find the inverse Laplace transform of

$$H(s) = \frac{30}{(s-3)^3\,(s^2+25)}$$

  - **Solution:** Split up $H(S) = F(s)G(s)$ where $F(s) = \frac{2}{(s-3)^3}$ and $G(s) = \frac{5}{s^2+25}$ so that

$$H(s) = 3 \cdot \frac{2!}{(s-3)^{2+1}} \cdot \frac{5}{s^2+5^2}$$

and since

$$\mathcal{L}^{-1}\{F\} = \mathcal{L}^{-1}\left\{\frac{2!}{(s-3)^{2+1}}\right\} = t^2 e^{3t},$$
$$\mathcal{L}^{-1}\{G\} = \mathcal{L}^{-1}\left\{\frac{5}{s^2+5^2}\right\} = \sin(5t)$$

so that

$$\begin{aligned}\mathcal{L}^{-1}\{H(s)\} &= 3\int_0^t f(t-\tau)g(\tau)d\tau \\ &= 3\int_0^t (t-\tau)^2 e^{3(t-\tau)}\sin(5\tau)\,d\tau\end{aligned}$$

but you also need to be prepared that one of the possible solutions is

$$\begin{aligned}\mathcal{L}^{-1}\{H(s)\} &= 3\int_0^t f(\tau)g(t-\tau)d\tau \\ &= 3\int_0^t \tau^2 e^{3\tau}\sin(5(t-\tau))\,d\tau.\end{aligned}$$

- **Example4:** Solve the IVP in terms of the convolution integrals:

$$4y'' + 4y' + 17y = g(t), \quad y(0) = 0, y'(0) = 0.$$

  - **Solution:**
  - **Step1:** Take $\mathcal{L}$ of both sides and solve $\mathcal{L}\{y\}$:

$$4\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) + 4\left(s\mathcal{L}\{y\} - y(0)\right) + 17\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

and plugging in the initial conditions we have

$$\mathcal{L}\{y\}\left(4s^2+4s+17\right) = \mathcal{L}\{g(t)\}$$

so that

$$\mathcal{L}\{y\} = \frac{\mathcal{L}\{g(t)\}}{4s^2+4s+17}$$

– **Step2:** Instead of doing partial fractions we will use the convolution integral. But first let us complete the square by first wrtiting

$$4s^2+4s+17=4\left(s^2+s+\frac{17}{4}\right)$$

hence we want add/subtract $\left(\frac{b}{2}\right)^2=\left(\frac{1}{2}\right)^2=\frac{1}{4}$hence

$$\begin{aligned}4\left(s^2+s+\frac{17}{4}\right)&=4\left(s^2+s+\frac{1}{4}-\frac{1}{4}+\frac{17}{4}\right)\\&=4\left(\left(s+\frac{1}{2}\right)^2+\frac{16}{4}\right)\\&=4\left(\left(s+\frac{1}{2}\right)^2+4\right)\end{aligned}$$

hence

$$\frac{\mathcal{L}\left\{g(t)\right\}}{4s^2+4s+17}=\frac{1}{4}\frac{1}{\left(\left(s+\frac{1}{2}\right)^2+4\right)}\mathcal{L}\left\{g(t)\right\}$$

– **Step3:** We take the inverse Laplace transform of

$$\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{\left(\left(s+\frac{1}{2}\right)^2+4\right)}\mathcal{L}\left\{g(t)\right\}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{4}\mathcal{L}\left\{f(t)\right\}\mathcal{L}\left\{g(t)\right\}\right\}$$

hence we need to take the inverse of

$$\begin{aligned}f(t)&=\mathcal{L}^{-1}\left\{\frac{1}{\left(\left(s+\frac{1}{2}\right)^2+4\right)}\right\}\\&=\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{\left(\left(s+\frac{1}{2}\right)^2+4\right)}\right\}\\&=\frac{1}{2}e^{-\frac{1}{2}t}\sin\left(2t\right).\end{aligned}$$

Thus using the formula $\mathcal{L}^{-1}\left\{F(s)G(s)\right\}=\left(f\star g\right)(t)=\int_0^t f\left(t-\tau\right)g\left(\tau\right)d\tau$ we have

$$\begin{aligned}y&=\mathcal{L}^{-1}\left\{\frac{1}{4}\mathcal{L}\left\{f(t)\right\}\mathcal{L}\left\{g(t)\right\}\right\}\\&=\frac{1}{4}\int_0^t f\left(t-\tau\right)g\left(\tau\right)d\tau\\&=\frac{1}{4}\int_0^t\frac{1}{2}e^{-\frac{1}{2}(t-\tau)}\sin\left(2\left(t-\tau\right)\right)g\left(\tau\right)d\tau\\&=\frac{1}{8}\int_0^t e^{-\frac{1}{2}(t-\tau)}\sin\left(2\left(t-\tau\right)\right)g\left(\tau\right)d\tau.\end{aligned}$$

- **Example5:** Compute the following integral

$$\int_0^5 e^{-x} \sin x dx$$

using only Laplace transforms.

  - Solution:: First we want to write this as a convolution:

$$\int_0^5 e^{-x} \sin x dx = e^{-5} \int_0^5 e^{5-x} \sin x dx.$$

and let

$$h(t) = \int_0^t e^{t-\tau} \sin \tau d\tau.$$

The Laplace trasnform of this is

$$\begin{aligned} \mathcal{L}\{h(t)\} &= \mathcal{L}\left\{\int_0^t e^{t-\tau} \sin \tau d\tau\right\} \\ &= \mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)d\tau\right\} \\ &= \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} \\ &= \mathcal{L}\left\{e^t\right\}\mathcal{L}\{\sin t\} \\ &= \frac{1}{(s-1)(s^2+1)} \end{aligned}$$

  - Now do partiial fractions on this and get

$$\frac{1}{(s-1)(s^2+1)} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{s+1}{s^2+1}\right)$$

  - Hence we can now take the inverse Laplace transform of this:

$$\begin{aligned} h(t) &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t. \end{aligned}$$

  - Thus we computed that

$$h(t) = \int_0^t e^{t-\tau} \sin \tau d\tau = \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t$$

  - Thus

$$\begin{aligned} e^{-5}\int_0^5 e^{5-x} \sin x dx &= e^{-5}h(5) \\ &= e^{-5}\left(\frac{1}{2}e^5 - \frac{1}{2}\cos 5 - \frac{1}{2}\sin 5\right). \end{aligned}$$

as needed

Sec. 7.1 - Systems of First Order Linear Equations

Predator-Prey System
Let $R(t)$ = prey population and let $F(t)$= predator population. Then the following is a system of first order equations:

$$\begin{aligned}\frac{dR}{dt} &= 2R - 1.2RF \\ \frac{dF}{dt} &= -F + 0.9RF.\end{aligned}$$

Notice that the prey and predator population are dependent on each other, and thus we need a system of equations.

Spring-Mass System

- Suppose we have mass attached to a spring which is attached to another mass attached to a spring.
- The behavior of one mass is affected the other (and vice versa)
- We need a system of ODE to solve such problems

Mixing Problem

- **Example 1:**
  - Salt water with concentration 3 g/L of salt flows into tank #1 at a rate 4 L/min. at a rate of 4 L/min.
    - The well mixed mixture from tank #1 flows into tank #2 at a rate of 4 L/min, and the well mixed mixture of tank #2 flows out at a rate of 4 L/min.
  - Tank #1 initially has 30 L of salt water with 6 g of salt dissolved in it.
  - Tank #2 initially has 20 L of fresh water.
  - Question: Write a system of ODEs representing this problem.
  - **Solution:**
  - **Step1**: First we define variables.
    - Let $x_1(t)$ and $x_2(t)$ be the amount of salt in tank #1 and tank #2, respectively at time t (minutes)
  - **Step2:** Use $x_i' = \text{Rate in } - \text{Rate out}$. We first get

$$\begin{aligned}x_1'(t) &= \begin{pmatrix}\text{concentration} \\ \text{of salt coming in}\end{pmatrix} \times \text{Rate} - \begin{pmatrix}\text{concentration} \\ \text{of salt coming out}\end{pmatrix} \times \text{Rate} \\ &= 3\frac{\text{g}}{L} \cdot 4\frac{\text{L}}{\text{min}} - \frac{x_1(t)}{\text{water in tank 1 @time } t} \cdot 4\frac{\text{L}}{\text{min}}\end{aligned}$$

but since

$$\text{water @ time } t = 30 + (4-4)\,t = 30$$

hence

$$x_1'(t) = 12 - \frac{4x_1(t)}{30}, \quad x_1(0) = 6.$$

- **Step3:** Use $x_i' =$ Rate in $-$ Rate out. We first get

$$x_2'(t) = \begin{pmatrix} \text{concentration} \\ \text{of salt coming in} \end{pmatrix} \times \text{Rate} - \begin{pmatrix} \text{concentration} \\ \text{of salt coming out} \end{pmatrix} \times \text{Rate}$$
$$= \frac{x_1(t)}{30} \frac{\text{g}}{L} \cdot 4\frac{\text{L}}{\text{min}} - \frac{x_2(t)}{20+(4-4)t} \cdot 4\frac{\text{L}}{\text{min}}$$
$$= \frac{4x_1(t)}{30} - \frac{4x_2(t)}{20}, \quad x_2(0) = 0$$

- Putting it together the system we have is

$$\begin{cases} x_1' = 12 - \frac{2}{15}x_1 & x_1(0) = 6. \\ x_2' = \frac{4}{30}x_1 - \frac{1}{5}x_2 & x_2(0) = 0 \end{cases}$$

- **Overview of system of ODES**
  - We will be dealing only with 2×2 systems. But in general we can have $n \times n$ systems.
  - The following is called a **linear homogeneous** system:

$$x_1' = a(t)x_1 + b(t)x_2$$
$$x_2' = c(t)x_1 + d(t)x_2$$

  - The following system is called **non-homogeneous** if

$$x_1' = a(t)x_1 + b(t)x_2 + g_1(t)$$
$$x_2' = c(t)x_1 + d(t)x_2 + g_2(t)$$

  where $g_1(t)$ or $g_2(t) \neq 0$.
  - As long as all the coefficient functions are all continuous then we have the existence and uniqueness of a solution $(x_1(t), x_2(t))$.
  - Any $2nd$ order ODE has a corresponding system of 2 equations.
    * That is if

$$a(t)y'' + b(t)y' + c(t)y = g(t)$$

    then we let $x_1 = y$ and $x_2 = y'$, and obtain

$$x_1' = x_2,$$
$$x_2' = -\frac{c}{a}x_1 - \frac{b}{a}x_2 + \frac{g}{a}.$$

- **Example1:** Turn

$$y'' + \frac{1}{2}y' + 2y = \sin t$$

into a system.
  - **Solution:**
  - Goal: We let $x_1 = y$, $x_2 = y'$ and set up the following system:

$$x_1' = ?$$
$$x_2' = ?$$

  - To do so, we start with what we defined and take derivatives:

$$\begin{array}{rcl} x_1 = y & \Longrightarrow & x_1' = y' = x_2 \\ x_2 = y' & \Longrightarrow & x_2' = y'' = -\frac{1}{2}y' - 2y - \sin t \end{array}$$

hence

$$\begin{aligned} x_2' &= -\frac{1}{2}y' - 2y - \sin t \\ &= -\frac{1}{2}x_2 - 2x_1 - \sin t, \text{ by definition} \end{aligned}$$

thus

$$\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - \frac{1}{2}x_2 + \sin t \end{cases}$$

- Is this Linear? Yes
- Is this homogeneous or non-homogenous? non-homogeneous because of the $\sin t$.

- How do we solve system of ODEs? one way is to turn them into 2nd order ODES
- **Example2:** Consider the following system of EQs

$$\begin{aligned} x_1' &= 3x_1 - 2x_2, \quad x_1(0) = 3 \\ x_2' &= 2x_1 - 2x_2, \quad x_2(0) = \frac{1}{2} \end{aligned}$$

Turn the following system into a single equation and solve for (x_1, x_2).

- **Solution:** To do so we solve for (x_1, x_2)one variable (the one that appears least often) using algebra:
  - **Step1:** From EQ1 solve for x_2:

$$\begin{aligned} 2x_2 = 3x_1 - x_1' &\implies x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1' \\ &\implies \boldsymbol{x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1'} \quad (\star) \end{aligned}$$

  - Take a serivative of both sides

$$x_2' = \frac{3}{2}x_1' - \frac{1}{2}x_1''$$

then set this equal to the RHS of EQ 2:

$$\frac{3}{2}x_1' - \frac{1}{2}x_1'' = 2x_1 - 2x_2 \quad (\star\star)$$

  - Plug $(\star)$ into $(\star\star)$:

$$\frac{3}{2}x_1' - \frac{1}{2}x_1'' = 2x_1 - 2\left(\frac{3}{2}x_1 - \frac{1}{2}x_1'\right)$$

and note that this is only dependent on the $x_1$variable. Doing some algebra we get the equation

$$x_1'' - x_1' - 2x_1 = 0 \quad (\star\star\star)$$

  - **Step2:** Now combining $(\star)$ into $(\star\star\star)$ we have

$$\begin{cases} x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1' & \Leftarrow \text{plug } x_1 \text{ here} \\ & \Uparrow \\ x_1'' - x_1' - 2x_1 = 0 & \implies \text{solve for } x_1 \end{cases}$$

· Solve x_1 using the methods from Chapter 3 and get

$$x_1 = c_1 e^{2t} + c_2 e^{-t}.$$

hence plug x_1 and its derivative x_1' into

$$\begin{aligned} x_2 &= \frac{3}{2}x_1 - \frac{1}{2}x_1' \\ &= \frac{3}{2}\left(c_1 e^{2t} + c_2 e^{-t}\right) - \frac{1}{2}\left(2c_1 e^{2t} - c_2 e^{-t}\right) \\ &= \frac{1}{2}c_1 e^{2t} + 2c_2 e^{-t}. \end{aligned}$$

which gives the two general solutions we needed.

- This method is tedious and we'll never use this method again.

Sec. 7.2 - Review of Matrices

We will start by considering the following **linear system with constant coefficients**:

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2. \end{aligned}$$

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a matrix and let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a vector we can define matrix-vector product as

$$A\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}.$$

- Give an example **matrix-vector** multiplication with $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ and $\boldsymbol{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
- Talk about **adding** vectors and **scaling** vectors!
- Thus we can write a system of ODES as

$$\boldsymbol{x}' = A\mathbf{x}$$

  where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.
  - A is the coefficient matrix
- **<u>Example:</u>** The system

$$\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= x_1 - 2x_2 \end{aligned}$$

  can be represent by

$$\begin{pmatrix} x_1' \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\boldsymbol{x}' = A\boldsymbol{x}$$

- **<u>Example:</u>** Verify that $\boldsymbol{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-4t}$ is a solution to

$$\boldsymbol{x}' = \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \boldsymbol{x}.$$

  - **<u>Solution:</u>** We plug $\boldsymbol{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-4t}$ into the LHS and RHS and see if the are equal to each other

$$\text{LHS} = \boldsymbol{x}' = \begin{pmatrix} e^{-4t} \\ -2e^{-4t} \end{pmatrix}' = \begin{pmatrix} -4e^{-4t} \\ 8e^{-4t} \end{pmatrix}$$

and

$$RHS = \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} e^{-4t} \\ -2e^{-4t} \end{pmatrix}$$
$$= \begin{pmatrix} 2e^{-4t} - 6e^{-4t} \\ 4e^{-4t} + 4e^{-4t} \end{pmatrix}$$
$$= \begin{pmatrix} -4e^{-4t} \\ 8e^{-4t} \end{pmatrix}$$

since

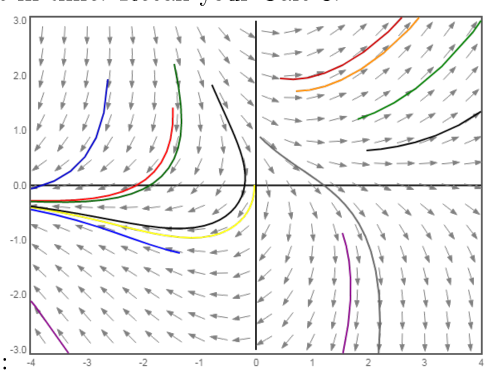
$$LHS = RHS$$

then $\boldsymbol{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-4t}$ is a solution to this system.

- **How to picture solutions to systems?**
  Phase Plane/Portrait:
  - What do solutions $\boldsymbol{x}(t)$ to $\boldsymbol{x}' = A\mathbf{x}$ look like? **They are parametric equations in the plane!** We graph in the x_1-x_2 plane since

    $$\boldsymbol{x}(t) = (x_1(t), x_2(t))$$

    is a vector (or point) that changes in time. Recall your Calc 3.

    * **Graph** would look like this:
    * This is called the **Phase Plane**
  - **Phase Portrait** are several phase planes for different initial conditions.
    * **Graph** a Phase Portrait
  - What are the simplest solutions?
    * An **equilibrum solution** are the constant solutions: $\boldsymbol{x}(t) = (x_0, y_0)$.
    * An **equilibrum solution** is a dot, since as times moves on it stays constant in the same place.
    * To solve for the equilibrium solution you set the derivative equalk to zero, in this case

      $$A\boldsymbol{x} = \mathbf{0}.$$
  - A d**irection field** can be drawn by drawing the following vector field

    $$\boldsymbol{F}(x_1, x_2) = A\boldsymbol{x}.$$

Sec. 7.3 - Systems of Linear Equatiosn: Linear Independence, Eigenvalues, Eigenvectors

Crash course in linear algebra:

Determinants:

- Suppose we want to find solutions to $A\mathbf{x} = \mathbf{0}$ or

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{array}{c} ax_1 + bx_2 = 0 \\ cx_1 + dx_2 = 0 \end{array}$$

- take

$$\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{array}{c} x_1 + 2x_2 = 0 \\ -x_1 + 3x_2 = 0 \end{array}$$

- Can we find solutions to this system? Yeah easy! its just algebra.
  - Notice that $\boldsymbol{x} = (x_1, x_2) = (0, 0)$ is always an equilibrium solution to

$$\begin{array}{c} ax_1 + bx_2 = 0 \\ cx_1 + dx_2 = 0 \end{array}.$$

  - When do we have nontrivial solutions? There is a way to knowing without actually soliving for it.
- The **determinant** of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined to be

$$\det A = ad - bc$$

- The **determinant** of a matrix $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ is defined to be

$$\det A = 3 - (-2) = 5$$

Theorem. *If A is matrix and $\det A \neq 0$ then the only solutions to the system $A\mathbf{x} = \mathbf{0}$ is the $(0,0)$, the origin.*

If $\det A = 0$ then there are infinitely many solutions.

- Note that $\det \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} = 0$ and hence there are nontrivial solutions.
  - **check** that $(-3, 1)$ is a nontrivial solution.
- If $\det A = 0$ are then A is **singular**, or **degenerate**.
  - If $\det A \neq 0$ are nonsingular, or nondegenerate, or **invertible**.
  - When A is invertible then the inverse matrix A^{-1} exists and we can solve

$$A\boldsymbol{x} = \boldsymbol{b} \quad (\star)$$

by

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}$$

and this is the **unique** solution to $(\star)$.

Independence:

- Another important concept in linear algebra is when two vectors are independent or dependent.
- If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ are vectors, then $c_1\mathbf{x} + c_2\mathbf{y}$ is said to be a **linear combination** of the two vectors.

  - Given **example**! The vector $\begin{pmatrix} -3 \\ 8 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$. Why? Because if we let $c_1 = 1$ and $c_2 = 2$ note that

$$1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$$

- **Definition:** The vectors $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ are said to be **linearly independent** if the only solutions to $c_1\mathbf{x} + c_2\mathbf{y} = 0$ are the trivial solutions $c_1 = c_2 = 0$.
  - A visual (and very usefull) way to understand this, two vectors are **linearly independent** if they do not lie in the same line through the origin.
  - Also, this means $\mathbf{x}$ and $\mathbf{y}$ are linearly independent if the only way $\mathbf{0}$ is a linear combination of these vectors, is the trivial linear combination.
  - **Multiple vectors?** The vectors $\boldsymbol{x}, \boldsymbol{y}, \mathbf{z}$ are linearly indepedent if the only solution to $c_1\mathbf{x} + c_2\mathbf{y} + c_3\boldsymbol{z} = 0$ are the trivial solutions $c_1 = c_2 = c_3 = 0$.
- **Example1:** Are $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ linearly independent? (yes)
  - Method1:

$$\begin{aligned} k_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 0 \quad &\iff \quad k_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 0 \\ &\iff \quad \begin{pmatrix} k_1 + 3k_2 \\ 2k_1 + 4k_2 \end{pmatrix} = 0 \\ &\iff \quad \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0 \\ &\iff \quad \det A = 4 - 6 \neq 0 \end{aligned}$$

    and from the theorem this tells us that they only solution is the trivial solution $k_1 = k_2 = 0$. Hence yes! **linearly independen**t!
  - Method2: Draw this on the $x - y$ plane and note they're NOT on the same line through the origin.
- **Example2:** Are $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 10 \end{pmatrix}$ linearly independent? (No) check both ways!
  - Method1:

$$\begin{aligned} k_1\begin{pmatrix} 3 \\ -5 \end{pmatrix} + k_2\begin{pmatrix} -6 \\ 10 \end{pmatrix} = 0 \quad &\iff \quad \begin{pmatrix} 3 & -6 \\ -5 & 10 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0 \\ &\iff \quad \det A = 30 - 30 = 0 \end{aligned}$$

    and from the theorem this tells us there are infinitely many solutions.
  - Hence **linearly dependent!**
  - Method2: Draw this on the $x - y$ plane and note they ARE on the same line through the origin.

Sec. 7.4 - Linearity principle

Linearity Principle:

Theorem. *Suppose $\boldsymbol{x}' = A\mathbf{x}$ is a linear system of differential equations.*
(1) If $\mathbf{x}(t)$ is a solution of this system and c is any constnat, then $c\mathbf{x}(t)$ is also a solution.
(2) If $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are two solutions of this systme, then $\mathbf{x}^{(1}(t) + \mathbf{x}^{(2)}(t)$ is also a solution.

- The point is we can create new solutions from one we already know are solutions via linear combinations!
- In fact, as long as I have one solution then I have infinitely many.
- In fact, we'll see that if as long as we have two solutions that are linearly indepedent, then we have al possible solutions.

Theorem. ***(The General Solution)*** *Suppose $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are solutions of the system $\boldsymbol{x}' = A\mathbf{x}$.* If $\mathbf{x}^{(1)}(0)$ and $\mathbf{x}^{(1)}(0)$ are linearly independent, then for any initial condition $\mathbf{x}(0) = (x_0, y_0)$, we can find constants c_1 and c_2 such that $\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(1)}(t)$ is the solution to the IVP

$$\boldsymbol{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

- This theorem says that as long as I find two linearly indepedent solutions $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ then every solution is of the form

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(1)}(t).$$

Sec. 7.5 - Basic Theory of Systems of 1st Order Linear EQs

Equilibrium solutions:

- Consider
$$\boldsymbol{x}' = A\mathbf{x}$$
where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.
- **Recall An equilibrium solution**, is $\mathbf{x}(t) = (x_0, y_0)$ such that $A\boldsymbol{x} = \mathbf{0}$. That is if
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff \begin{pmatrix} ax + by = 0 \\ cx + dy = 0 \end{pmatrix}.$$
These are the constant solutions.
  - We usually assume $\det A \neq 0$ so that $\boldsymbol{x}(t) = \mathbf{0}$ is the only equilibrium solution.
  - These are the constant solutions.
  - We will ask if other solutions are stable or unstable? That is, do other solutions approach the origin or not.

Some Linear Algebra

- Draw a vector field, with some straight line solutions
  - An **eigenvector** is a vector where the vector field points in the same or opposite direction as the vector itself.

Definition. Given a matrix A, a number λ is called an **eigenvalue** of A if there is a nonzero vector $\mathbf{v}$ such that
$$A\mathbf{v} = \lambda\mathbf{v}.$$
The corresponding vector $\mathbf{v}$ is called an **eigenvector of the eigenvalue λ.**

Derivation:

- Our goal is to first find eigenvalue, then find the corresponding eigenvector:

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ identity matrix then $\lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Note that by the definition of an eigenvalue, we must have that for some $\mathbf{v}$

$$\begin{aligned} A\mathbf{v} = \lambda\mathbf{v} &\iff A\mathbf{v} = \lambda\mathbf{v} \\ &\iff A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \\ &\iff A\mathbf{v} - \lambda I\mathbf{v} = \mathbf{0} \\ &\iff (A - \lambda I)\,\mathbf{v} = \mathbf{0}. \end{aligned}$$

Now $A - \lambda I$ is actually another matrix. What matrix? Let's see

$$\begin{aligned} A - \lambda I &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}. \end{aligned}$$

So using our theorem from a previous section , we know when the equation $(A - \lambda I)\,\mathbf{v} = \mathbf{0}$ has nontrivial solution? Recall from the theorem that if

$$\det(A - \lambda I) = 0$$

then we had nontrivial solutions!! Let's solve for λ, because we know how to find determinants!!!

$$\begin{aligned}\det(A-\lambda I)=0 &\iff \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}=0\\ &\iff (a-\lambda)(d-\lambda)-bc=0.\\ &\iff (\text{something})\,\lambda^2+(\text{something})\lambda+(\text{something})=0\end{aligned}$$

so solve for λ and that will be your eigenvalue! This polynomial is called the **characteristic polynomial!**

<u>Example1:</u> Find the eigenvalues and the corresponding eigenvectors of $A=\begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}$.

- **<u>Step1:</u>** Solve

$$\begin{aligned}\det(A-\lambda I)=0 &\iff \det\begin{pmatrix} 2-\lambda & 3 \\ 0 & -4-\lambda \end{pmatrix}=0\\ &\iff (2-\lambda)(-4-\lambda)-0\cdot 3=0.\\ &\iff (2-\lambda)(-4-\lambda)=0\\ &\iff \lambda=-4,2.\end{aligned}$$

- **<u>Step2:</u>** Find the eigenvectors: Let's start with $\lambda_1=2$ then $\mathbf{v}_1=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is an eigenvector if

$$\begin{aligned}A\mathbf{v}=2\mathbf{v} &\iff \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\\ &\iff \begin{pmatrix} 2x_1+3x_2 \\ -4x_2 \end{pmatrix}=2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\\ &\iff \begin{cases} 2x_1+3x_2=2x_1 \\ -4x_2=2x_1 \end{cases}\\ &\iff x_2=0 \text{ and } x_1=\text{can be anything.}\end{aligned}$$

  so choose $\mathbf{v}_1=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as the eigenvector.

- **<u>Step3:</u>** Find the eigenvectors: Now let $\lambda_2=-4$ then $\mathbf{v_2}=\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is an eigenvector if

$$\begin{aligned}A\mathbf{v}=-4\mathbf{v} &\iff \begin{cases} 2x_1+3x_2=-4x_1 \\ -4x_2=-4x_2 \end{cases}\\ &\iff \begin{cases} 6x_1+3x_2=0 & \Longrightarrow x_2=-2x_1 \\ -4x_2=-4x_2 \end{cases}\end{aligned}$$

  so choose $\mathbf{v}_2=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ as the eigenvector.

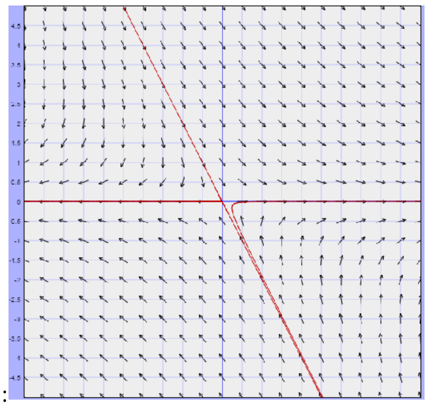
  – ($\star$)Notice that any multiple would also be an eigevnvector. So we can also choose

$$\mathbf{v}_2=-3\begin{pmatrix} 1 \\ -2 \end{pmatrix}=\begin{pmatrix} -3 \\ 6 \end{pmatrix}.$$

<u>Back to Differential Equations:</u> Why Eigenvectors?

- Consider

$$\boldsymbol{x}' = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \mathbf{x}$$



- The direction field looks like this:
- We want to search for **straight line solutions** (Note them on the graph)
  - Because they probably have easy explicit formulas and more importantly they are linearly independent! (why?)
- How do we find them?
  - From the geometry of the phase plane. If $\mathbf{x}$ is a straight line solution, then notice that $A\mathbf{x} = \lambda\mathbf{x}$ for some λ. Because the vector $A\mathbf{x}(x,y)$ points in the same direction as the vector from $(0,0)$ to $\mathbf{x}(x,y)$. This means the solution $\mathbf{x}$ are all **eigenvectors**!!!!!!!
  - So if we can find an eigenvector and its eigenvalue then we would have found a straight line solution.

Claim. Suppose λ is an eigenvalue and $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ is an eigevenvector. Then we claim that

$$\mathbf{x}(t) = e^{\lambda t} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{\lambda t} x \\ e^{\lambda t} y \end{pmatrix}$$

is a straight line solution:

Proof. We just have to check that LHS and RHS equal for

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

The Left Hand Side is

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt} \begin{pmatrix} e^{\lambda t} x \\ e^{\lambda t} y \end{pmatrix} = \begin{pmatrix} \lambda e^{\lambda t} x \\ \lambda e^{\lambda t} y \end{pmatrix} = \lambda \begin{pmatrix} e^{\lambda t} x \\ e^{\lambda t} y \end{pmatrix} = \lambda\mathbf{x}$$

and the Right Hand Side is

$$A\mathbf{x} = Ae^{\lambda t}\mathbf{v} = e^{\lambda t} A\mathbf{v} = e^{\lambda t}\lambda\mathbf{v} = \lambda\left(e^{\lambda t}\mathbf{v}\right) = \lambda\mathbf{x}.$$

So yes they are equal! □

Example2: Find the General solution:
Going back to our example

$$\frac{d\boldsymbol{x}}{dt} = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} \mathbf{x}.$$

recall that we have the eigenvalue $\lambda_1 = 2$ with eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $\lambda_2 = -4$ with eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. From the theorem we have just proved: Then we know two straight line solutions:

$$\mathbf{x}^{(1)}(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{x}^{(2)}(t) = e^{-4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

If these are independent, then we can form the general solution using a theorem from last time! Are $\mathbf{Y}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{Y}_2(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ independent?

Take the Wronkian:

$$W\left[\mathbf{x}^{(1)}, \boldsymbol{x}^{(2)}\right] = \begin{vmatrix} e^{2t} & e^{-4t} \\ 0 & -2e^{-4t} \end{vmatrix} = -2e^{-2t} \neq 0$$

hence $\left\{\mathbf{x}^{(1)}, \boldsymbol{x}^{(2)}\right\}$ forms a **fundamental set of solutions**.

- Yes! remenber that these vector are in completely different lines! Therefore they are independent!

Thus the general solution is

$$\mathbf{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

- **Summary:**

Theorem. *Suppose A is a matrix with distinct, real eigenvalues λ_1, λ_2 with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, repectively. Then the solutions **general solution** of the system*

$$\boldsymbol{x}' = A\mathbf{x}$$

is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

And the solutions $\mathbf{x}^{(1)}(t) = e^{\lambda_1 t}\mathbf{v}_1, \mathbf{x}^{(2)}(t) = e^{\lambda_2 t}\mathbf{v}_2$ are linearly independent.

If enough time teach the following tricks:

- Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let's figure out how to use this.
- $\mathrm{tr}A = a + d$. the sum of the diagonal.
- **Fact 1:** the eigenvalues add to the trace ,$\lambda_1 + \lambda_2 = a + d$.
- **Fact 2:** if the two rows add to the same number $a + b = a + d$, then that number is an eigenvalue with eigenvector $\mathbf{v} = (1, 1)$
- **Fact 3:** if the two columns add to the same number $a + c = b + d$, then that number is an eigenvalue and an eigenvector for the **OTHER** eigenvalue is $(1, -1)$

Example3:

- Take $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$.
  - Using Fact 2: $2 + 2 = 1 + 3 = 4$.So $\lambda_1 = 4$ is an eigenvalue and $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the eigenvalue.
  - Using Fact 1:

$$\begin{aligned} \lambda_1 + \lambda_2 &= 2 + 3 = 5 \\ \lambda_2 &= 5 - 4 = 1 \end{aligned}$$

– So we only need to the find $\mathbf{v}_2$. Need actual work here:

$$\begin{cases} 2x_1 + 2x_2 = x_1 \\ x_1 + 3x_2 = x_2 \end{cases} \iff \begin{cases} x_1 + 2x_2 = 0 \\ x_1 + 2x_2 = 0 \end{cases}$$

$$\iff x_1 = -2x_2$$

$$\iff \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

<u>Example4:</u>

- Take $A = \begin{pmatrix} 4 & 5 \\ 1 & 8 \end{pmatrix}$.
  - $\lambda_1 = 9$ and $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
  - Since $\lambda_1 + \lambda_2 = 12$ then $\lambda_2 = 3$.

Sec. 7.5(b) - Phase Portraits for system w/real eigenvalues

Here are two facts we may have not noticed last time:

- If the eigenvalue λ is negative, the straight line solution $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ tends to the origin as $t \to \infty$. Draw a picture with example $\mathbf{x}(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
- If the eigenvalue λ is positive, the straight line solution $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ tends to away from the origin as $t \to \infty$. Draw a picture with example $\mathbf{x}(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Saddles:

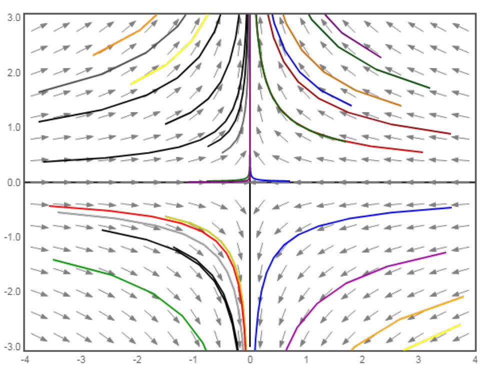
Definition. A linear system for which we have one positive and one negative eigenvalue has an equilibrium point that is called a **saddle**.

Example1: Consider

$$\boldsymbol{x}' = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x}$$

if we solve this system we get

$$\mathbf{x}(t) = c_1 e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Draw straight line solutions in Phase Plane
- Draw other solutions in Phase Plane. Show this by using $t \to \infty$ analysis.
- 

Make an analysis on the $x(t), y(t)$ graphs with initial point $(1, 1)$ and get **graph** that looks like this::

Example2: Consider

$$\boldsymbol{x}' = \begin{pmatrix} 8 & -11 \\ 6 & -9 \end{pmatrix}\mathbf{x}$$

using trick about eigenvalues we get that $\lambda_1 = -3$ ad $\lambda_2 = 2$ with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$.
if we solve this system we get

$$\mathbf{x}(t) = c_1 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 11 \\ 6 \end{pmatrix}.$$

- Draw straight line solutions in Phase Plane
- Draw other solutions in Phase Plane. Show this by using $t \to \infty$ analysis.
- Also draw $x(t), y(t)$ curves for the following initial conditions:
  - $(0, -5)$

– (20, 10) should be inside one of the parabola thingys

Asymptotically stable node

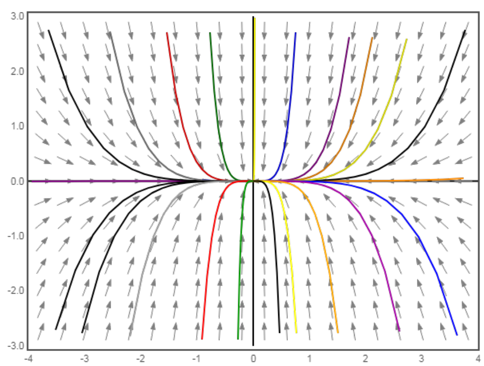
Definition. A linear system for which we have both negative, distinct eigenvalues has an equilibrium point that is called a **asymptotically stable node.**

Example1: Consider

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \mathbf{x}$$

if we solve this system we get

$$\mathbf{x}(t) = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-4t} \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Draw straight line solutions in Phase Plane with ARROWs
- Draw other solutions in Phase Plane.
  - MAKE DISTINCTION ON THE ARROWS.(They **sink** in)
- 

Example2: Consider

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -2 \\ 1 & -3 \end{pmatrix} \mathbf{x}$$

if we solve this system we get

$$\mathbf{x}(t) = c_1 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Asymptotically unstable node

Definition. A linear system for which we have both positive, distinct eigenvalues has an equilibrium point that is called a **asymptotically unstable node.**

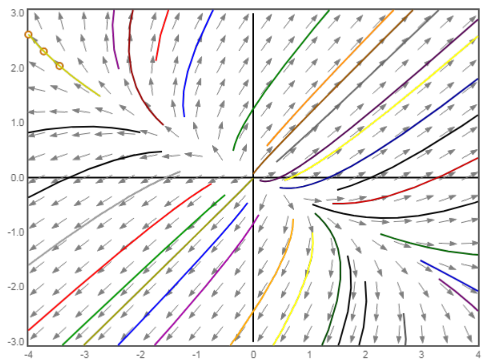
Example1: Consider

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

if we solve this system we get

$$\mathbf{x}(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

- Draw straight line solutions in Phase Plane with ARROWs
- Draw other solutions in Phase Plane.
  - MAKE DISTINCTION ON THE ARROWS.(They **SOURCE** out)



- **Pictures tell us more than intuition:**
  If there's any more time draw **Phase Portraits** for the following countivive system (profit example)

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix} \mathbf{x}$$

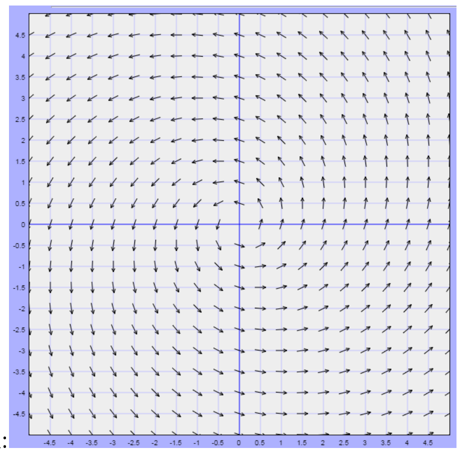
use tricks and get

$$\mathbf{x}(t) = k_1 e^t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + k_2 e^{-5} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

SEC. 7.6- COMPLEX EIGENVALUES

- We have only worked when we have distinct real eigenvectors.
- But what if we get complex numbers as eigenvectors?
  - When will this even happen?
- Consider

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \mathbf{x}.$$

- With this direction field: 
- The intuition form before doesn't work. There are no straight line solutions. It looks like solutions will be spirals.
- So we shall proceed as have done before, but this time you will see that we will have complex numbers

Complex numbers:

- Here are some facts. Complex numbers are of the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.
  - $i^2 = -1$. Remember this.
- Also we'll need to know **Euler's Formula:** $e^{ib} = \cos b + i \sin b$. So

$$e^{a+ib} = e^a e^{ib} = e^a (\cos b + i \sin b) = e^a \cos b + i e^a \sin b.$$

Example1: Find general solutions of

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \mathbf{x}.$$

- Find the eigenvalue:

$$\begin{aligned} \det(A - \lambda I) = 0 \quad &\Longleftrightarrow \quad (1-\lambda)^2 + 9 = 0 \\ &\Longleftrightarrow \quad \lambda = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm 3i. \end{aligned}$$

- **Pick one eigenvalue:** Find the eigenvector: Solve

$$\begin{cases} x_1 - 3x_2 = (1+3i)\, x_1 \\ 3x_1 + x_2 = (1+3i)\, x_2 \end{cases}$$

One of these equations will be redundant. So pick one equation to find the eigenvector

$$\begin{aligned} 3x_1 + x_2 = (1+3i)\, x_2 \quad &\Longleftrightarrow \quad 3x_1 = 3ix_2 \\ &\Longleftrightarrow \quad x_1 = ix_2 \\ &\Longleftrightarrow \quad \text{pick } x_2 = 1 \text{ and get } x_1 = i. \end{aligned}$$

So the eigenvector $\mathbf{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ is associated with the eigenvalue $\lambda = 1 + 3i$.

- Find the corresponding **complex solution:**

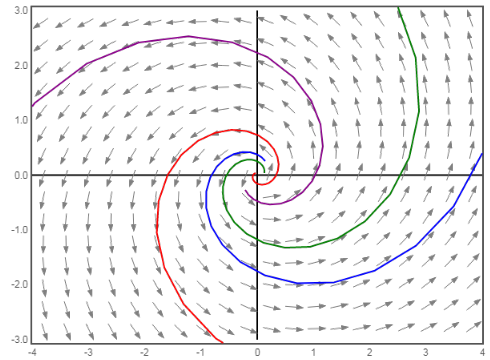
$$\begin{aligned} \mathbf{x}_{comp}(t) &= \mathbf{v}e^{\lambda t} \\ &= \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(1+3i)t} \\ &= \begin{pmatrix} i \\ 1 \end{pmatrix} e^{t} e^{3ti} \\ &= \begin{pmatrix} i \\ 1 \end{pmatrix} e^{t} (\cos 3t + i \sin 3t) \\ &= \begin{pmatrix} ie^{t} (\cos 3t + i \sin 3t) \\ 1e^{t} (\cos 3t + i \sin 3t) \end{pmatrix} \\ &= \begin{pmatrix} ie^{t} \cos 3t - e^{t} \sin 3t \\ e^{t} \cos 3t + ie^{t} \sin 3t \end{pmatrix} \\ &= \begin{pmatrix} -e^{t} \sin 3t + ie^{t} \cos 3t \\ e^{t} \cos 3t + ie^{t} \sin 3t \end{pmatrix} \textbf{put } i\textbf{'s together} \\ &= \begin{pmatrix} -e^{t} \sin 3t \\ e^{t} \cos 3t \end{pmatrix} + i \begin{pmatrix} e^{t} \cos 3t \\ e^{t} \sin 3t \end{pmatrix} \\ &= \mathbf{x}_{re}(t) + i\mathbf{x}_{im}(t). \end{aligned}$$

Theorem. *If $\mathbf{x}_{comp}(t) = \mathbf{x}_{re}(t) + i\mathbf{x}_{im}(t)$ is a solution to the linear system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$. Then $\mathbf{x}_{re}(t), \mathbf{x}_{im}(t)$ are two linearly independent solutions to the system as well.*

- So it turns out the general solution to this system is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -e^{t} \sin 3t \\ e^{t} \cos 3t \end{pmatrix} + c_2 \begin{pmatrix} e^{t} \cos 3t \\ e^{t} \sin 3t \end{pmatrix}$$

  - REMOVE THE i!!!!!
- **Qualitative Analysis:** We already know from the direction field that the curves are going to be spirals. The e^t terms says that the solutions are getting father and father away from the origin:
  - Thus **graph** of the Phase Portrait looks like this:



Summarize:

Theorem. *If a linear system as eigenvalue $\lambda = \alpha \pm \beta i$. Then the solution curves from spirals about the origin with natural period $\frac{2\pi}{\beta}$, natural frequency $\frac{\beta}{2\pi}$ and*

(1) If $\alpha < 0$, it's a asymptotically stable spiral point.
(2) If $\alpha > 0$, its a asymptotically unstable spiral point.
(3) If $\alpha = 0$, its a center.

So the previous example was a **asymptotically unstable spiral**. Let's see other examples.

Example2: A center

We convert $\frac{d^2y}{dt^2} = -2y$ into

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \mathbf{x}$$

and get characterisitc polynomial of $\lambda^2 + 2$ and get $\lambda = \pm i\sqrt{2}$.

- Phase Portrait:
  - The **period** is $\frac{2\pi}{\sqrt{2}}$ and **frequency** is $\frac{\sqrt{2}}{2\pi}$.
  - One thing the theorem doesn't tell us is if the spirals are clockwise or counter-clockwise.
  - So figure out the vector image at $(1,0)$ and $(0,1)$ and get $A(1,0) = (0,-2)$ and $A(0,1) = (1,0)$. This is thus **clockwise.**
  - Since $\alpha = 0$ then this is center and a **graph** looks like:
- The solution would be

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos\sqrt{2}t \\ -\sqrt{2}\sin\sqrt{2}t \end{pmatrix} + c_2 \begin{pmatrix} \sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t \end{pmatrix}.$$

- The solutions are ellipese. By we don't know if the major and minor axis are in the y and x axis.

Example3: **asymptotically stable spiral**.

Suppose

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \mathbf{x}$$

and get characterisitc polynomial that $\lambda^2 + 4\lambda + 13 = 0$ and get $\lambda = -2 \pm 3i$.

- Phase Portrait:
  - The **period** is $\frac{2\pi}{3}$ and **frequency** is $\frac{3}{2\pi}$.
  - One thing the theorem doesn't tell us is if the spirals are clockwise or counter-clockwise.

  - So figure out the vector for at $(1,0)$ and $(0,1)$ and get $A(1,0)=(-2,3)$ and $A\,(0,1)=(-3,-2)$. This is thus **counterclockwise.**
  - Since $\alpha=-2<0$ then this is **spiral sink.**
- The solution would be for $\lambda=-2+3i$ get eigenvector of $\mathbf{v}=(i,1)$ and solution of

$$\mathbf{x}(t)=c_1\begin{pmatrix} -e^{-2t}\sin 3t \\ e^{-2t}\cos 3t \end{pmatrix}+c_2\begin{pmatrix} e^{-2t}\cos 3t \\ e^{-2t}\sin 3t \end{pmatrix}.$$

Sec. 7.8 - Repeated and zero eigenvalues

Repeated roots:
Suppose

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix} \mathbf{x}.$$

Let's first the the **eigenvalues**:

$$\begin{aligned} \det \begin{pmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{pmatrix} = 0 \quad &\iff \quad (1-\lambda)(5-\lambda) + 4 = 0 \\ &\iff \quad \lambda^2 - 6\lambda + 9 = 0 \\ &\iff \quad (\lambda - 3)^2 = 0, \\ &\iff \quad \lambda = 3. \end{aligned}$$

So it turns out that we will guess. Our guess is going to be similar to the guess to what we did in a previous section. The solution will be of the form:

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}_0 + te^{\lambda t}\mathbf{v}_1.$$

where $\mathbf{Y}(0) = \mathbf{v}_0$ is the initial condition. So what will $\mathbf{v}_1$ be? Let's figure it out. For this solution to work we check that the LHS equals to RHS for

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

for this specific $\mathbf{Y}$.

The **LHS**: Do calculus like we would normally do:

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \lambda e^{\lambda t}\mathbf{v}_0 + e^{\lambda t}\mathbf{v}_1 + \lambda te^{\lambda t}\mathbf{v}_1 \\ &= (\lambda\mathbf{v}_0 + \mathbf{v}_1) e^{\lambda t} + (\lambda\mathbf{v}_1) te^{\lambda t} \end{aligned}$$

The RHS: We have

$$\begin{aligned} A\mathbf{x} &= A\left(e^{\lambda t}\mathbf{v}_0 + te^{\lambda t}\mathbf{v}_1\right) \\ &= A\mathbf{v}_0 e^{\lambda t} + (A\mathbf{v}_1) te^{\lambda t} \end{aligned}$$

So matching the coefficients we get that we must have

$$\lambda\mathbf{v}_1 = A\mathbf{v}_1 \text{ and } \lambda\mathbf{v}_0 + \mathbf{v}_1 = A\mathbf{v}_0.$$

This means either $\mathbf{v}_1$ is an eigenvector or the zero vector. We also have that we can explicitely get

$$\begin{aligned} \lambda\mathbf{v}_0 + \mathbf{v}_1 = A\mathbf{v}_0 \quad &\iff \quad \mathbf{v}_1 = A\mathbf{v}_0 - \lambda\mathbf{v}_0 \\ &\iff \quad \mathbf{v}_1 = (A - \lambda I)\mathbf{v}_0. \end{aligned}$$

Thus we have the following theorem:

Theorem. *Suppose $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ is a system with λ being a double root. Then the general solution is of the form:*

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}_0 + te^{\lambda t}\mathbf{v}_1$$

where $\mathbf{v}_0$ in the initial condition and

$$\mathbf{v}_1 = (A - \lambda I)\mathbf{v}_0.$$

If $\mathbf{v}_1 \neq \mathbf{0}$ then it is an eigenvector, and if $\mathbf{v}_1 = 0$ then $\mathbf{v}_0$ is an eigenvector and $\mathbf{Y}(t)$ is a striaght line solution.

- **(Warning)** Never think that $e^{\lambda t}\mathbf{v}_0, te^{\lambda t}\mathbf{v}_1$ are solution by their own. It doesn't work that way here. This is a completely different way to solving this problem.

So what do we do? Can we find some eigenvectors:

Example1:

We consider

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}\mathbf{x}$$

with initial condition $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

- **Step1:** Recall we found $\lambda = 3$ to be an eigenvalue. Our theorem gives us that $\mathbf{v}_1$ will be an eigenvector. But we can find it directly by putting $\lambda = 3$ into

$$\det\begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} = 0$$

  and getting $x = -y$ so $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector.

- **Step2:** Suppose $\mathbf{v}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is the initial condition then what is is $\mathbf{v}_1$?

$$\begin{aligned} \mathbf{v}_1 &= (A - \lambda I)\,\mathbf{v}_0 \\ &= \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} -2x_0 - 2y_0 \\ 2x_0 + 2y_0 \end{pmatrix}. \end{aligned}$$

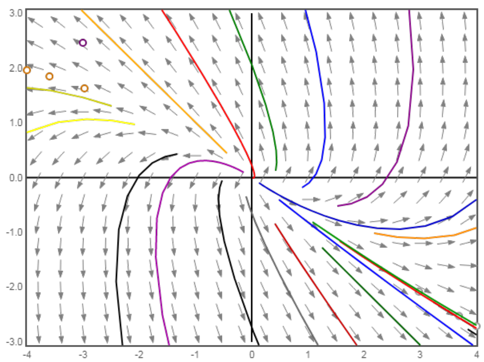
- **Step3:** Write the solution

$$\begin{aligned} \mathbf{x}(t) &= e^{3t}\mathbf{v}_0 + te^{3t}\mathbf{v}_1 \\ &= e^{3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{\lambda 3t}\begin{pmatrix} -2x_0 - 2y_0 \\ 2x_0 + 2y_0 \end{pmatrix} \end{aligned}$$

  and pluggin the inital condition we have

$$\mathbf{x}(t) = e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + te^{3t}\begin{pmatrix} -6 \\ 6 \end{pmatrix}.$$

- **Step4:** We plot the solutions by plotting the straight line solutions first ($y = -x$) and then making the following **graph** (an almost spiral, but a **asymptotically unstable improper node**) putting outwards (because $\lambda = 3 > 0$) (still check clockwise/counterclockwise):



Example 2
For

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{x}$$

and get $\lambda = -2$ and get

$$\mathbf{Y}(t) = e^{-2t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-2t} \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$

wih the following **graph** (note the sink):

- This would be an **asymptotically stable improper node** because $\lambda < 0$.

Every vector is an eigenvector:
Suppose

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mathbf{Y}$$

then $\lambda = a$ and then every vector is an eigenvector. Because every vector is an eigenvector, then every solution is a ray the either aproaches or leaces the origin. Here is a **graph**:
Systems with a zero eigenvalue:
Example1:
Notice that when $\det A = 0$ then from before must have infinitely many equilbrium solutions. Notice for all the other examples we always had the origin as the equilibrium point. So when $\det A = 0$ then we know that there exists

$$\lambda_1 = 0.$$

Consider

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \mathbf{x}.$$

Since $\det A = 0$ then this will always tell us that one of the eigenvalues is $\lambda_1 = 0$. But since $\text{tra}A = -4$ and using the fact that $\lambda_1 + \lambda_2 = -4$ then $\lambda_2 = -4$.

- The eigenvector for $\lambda_1 = 0$ is the line $y_1 = 3x_1$ so choose $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
- The eigenvector for $\lambda_1 = -4$ is the line $x_2 = -y_2$ so choose $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

- The solution is like before in the discussion of distinct eigenvalues:

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{0t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \end{aligned}$$

- First thing: There is a line of equilibrium solutions at $y = 3x$.
- All other solutions are iether point outwards or inwards. Here inwards becasue of the $\lambda = -4$
- Here is a **graph:**

Sec. 7.9 - Non-homegeneous Linear Equations

- Suppose we have a non-homogeneous system

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{g}(t)$$

  where $\boldsymbol{g}(t) \neq \mathbf{0}$.
- Methods:
  - diagonalization
  - Laplace transforms
  - variation of parameters
  - method of undetermined coefficients - we'll learn this today
    * A is constant
    * $\boldsymbol{g}(t)$ consists or polynomial, sin, cos, or exp.
- MOUC:
  - **Step1:** Find $\boldsymbol{x}_c$ which is the general solution to the homogeneous equation

$$\boldsymbol{x}' = A\boldsymbol{x}.$$

  - **Step2:** Make guess for $\boldsymbol{x}_p$, which is a particular solution to

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{g}(t)$$

    based on $\boldsymbol{g}(t)$.
  - **Step3:** Adjust if there are any repeats with $\boldsymbol{x}_c$. By multiplying by t.
- **Example1:** Find general solution of

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 2e^t \\ -e^t \end{pmatrix}.$$

  - **Solution:**
  - **Step1:** Find $\boldsymbol{x}_c$: which solves

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \boldsymbol{x}$$

    * Eigenvalue $\lambda_1 = -2$ has an eigenvector $\boldsymbol{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$
    * Eigenvalue $\lambda_2 = -1$ has an eigenvector $\boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.
    * So the homogeneous solution is

$$\boldsymbol{x}_c = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}.$$

  - **Step2:** We make our first guess $\boldsymbol{x}_p$:based on $\boldsymbol{g}(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$
    * 1st guess: $\boldsymbol{x}_p = \boldsymbol{a}e^t = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t$ where $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is some vector.
      · Since $\boldsymbol{a}e^t$ is not part of $\boldsymbol{x}_c$ then we made the right guess.

- **Step3:** Then plug $\boldsymbol{x}_p$ into the ODE:

$$\boldsymbol{x}_p' \stackrel{?}{=} A\boldsymbol{x_p} + \boldsymbol{g}(t) \iff \frac{d}{dt}\left(\boldsymbol{a}e^t\right) \stackrel{?}{=} A\boldsymbol{a}e^t + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$$
$$\iff \boldsymbol{a}e^t \stackrel{?}{=} A\boldsymbol{a}e^t + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^t$$
$$\iff \boldsymbol{a} \stackrel{?}{=} A\boldsymbol{a} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
$$\iff -\begin{pmatrix} 2 \\ -1 \end{pmatrix} \stackrel{?}{=} A\boldsymbol{a} - \boldsymbol{a}$$
$$\iff \begin{pmatrix} -2 \\ 1 \end{pmatrix} \stackrel{?}{=} (A - I)\,\boldsymbol{a}$$

and since $A - I = \begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix}$ then we need to solve

$$\begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

  * You can row reduce or solve using regular algebra to obtain

$$a_1 = \frac{7}{6}, a_2 = -\frac{5}{6}$$

  * Thus the general solution is

$$\boldsymbol{x}(t) = c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-2t} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{6}\begin{pmatrix} 7 \\ -5 \end{pmatrix} e^t.$$

- **Example2:** Find general solution of

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 2e^{-2t} \\ -e^{-2t} \end{pmatrix}.$$

  - **Solution:**
  - **Step1:** Find $\boldsymbol{x}_c$:which is the same as Example 1:
    * So the homogeneous solution is

$$\boldsymbol{x}_c = c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-2t} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}.$$

  - **Step2:** We make our first guess $\boldsymbol{x}_p$:based on $\boldsymbol{g}(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-2t}$
    * 1st guess: $\boldsymbol{x}_p = \boldsymbol{a}e^{-2t} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^t$ where $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is some vector.
      · Since $\boldsymbol{a}e^{-2t}$ is part of $\boldsymbol{x}_c$ then we need to reguess
    * 2ns guess: $\boldsymbol{x}_p = \boldsymbol{a}te^{-2t} + \boldsymbol{b}e^{-2t}$ (**note that this is different than in the non system case**)

– **Step3:** Then plug $\boldsymbol{x}_p$ into the ODE:

$$\begin{aligned} RHS &= A\boldsymbol{x_p} + \boldsymbol{g}(t) \\ &= A\boldsymbol{a}te^{-2t} + A\boldsymbol{b}e^{-2t} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-2t} \\ &= A\boldsymbol{a}te^{-2t} + \left[A\boldsymbol{b} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right] e^{-2t} \end{aligned}$$

while the LHS is equal to

$$\begin{aligned} LHS &= \frac{d}{dt}\left[\boldsymbol{a}te^{-2t} + \boldsymbol{b}e^{-2t}\right] = \boldsymbol{a}e^{-2t} - 2\boldsymbol{a}te^{-2t} - 2\boldsymbol{b}e^{-2t} \\ &= -2\boldsymbol{a}te^{-2t} + (\boldsymbol{a} - 2\boldsymbol{b})\, e^{-2t}. \end{aligned}$$

– Collecting terms we have:

$$\begin{aligned} te^{-2t}\text{term} :& -2\boldsymbol{a} = A\boldsymbol{a} \qquad (\star) \\ e^{-2t}\text{term} :& \boldsymbol{a} - 2\boldsymbol{b} = A\boldsymbol{b} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad (\star\star) \end{aligned}$$

– Using $(\star)$ we have

$$\begin{aligned} -2\boldsymbol{a} = A \iff& (A + 2I)\, \boldsymbol{a} = \boldsymbol{0} \\ \iff& \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

hence $a_1 = -\frac{1}{2}a_2$ or

$$\boldsymbol{a} = \begin{pmatrix} -\frac{1}{2}r \\ r \end{pmatrix}.$$

– Back to $(\star\star)$ we have

$$\begin{aligned} \boldsymbol{a} - 2\boldsymbol{b} = A\boldsymbol{b} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} &\iff \\ A\boldsymbol{b} + 2\boldsymbol{b} = \boldsymbol{a} - \begin{pmatrix} 2 \\ -1 \end{pmatrix} &\iff \\ (A + 2I)\, \boldsymbol{b} = \begin{pmatrix} -\frac{1}{2}r - 2 \\ r + 1 \end{pmatrix} &\iff \\ \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}r - 2 \\ r + 1 \end{pmatrix} &\iff \\ \left[\begin{array}{cc|c} 2 & 1 & -\frac{1}{2}r - 2 \\ -2 & -1 & r + 1 \end{array}\right] \iff \left[\begin{array}{cc|c} 2 & 1 & -\frac{1}{2}r - 2 \\ 0 & 0 & \frac{1}{2}r - 1 \end{array}\right] & \end{aligned}$$

and this is consistence if and only if

$$\frac{1}{2}r - 1 = 0 \implies r = 2.$$

– Thus

$$\begin{aligned} 2b_1 + b_2 = -\frac{1}{2}2 - 1 \iff& 2b_1 + b_2 = -3 \\ \iff& b_2 = 3 - 2b_1 \end{aligned}$$

hence

$$\boldsymbol{b}=\begin{pmatrix} b_1 \\ 3-2b_1 \end{pmatrix}$$

and since b_1 can be anything choose $b_1=0$ to get

$$\boldsymbol{b}=\begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

– Recall

$$\boldsymbol{a}=\begin{pmatrix} -\frac{1}{2}r \\ r \end{pmatrix}=$$

– **<u>Step4:</u>** Recall that

$$\begin{aligned}\boldsymbol{x}(t)&=\boldsymbol{x}_c+\boldsymbol{x}_p\\&=c_1\begin{pmatrix} -1 \\ 2 \end{pmatrix}e^{-2t}+c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{-t}\\&+\begin{pmatrix} -1 \\ 2 \end{pmatrix}te^{-2t}+\begin{pmatrix} 0 \\ 3 \end{pmatrix}e^{-2t}\end{aligned}$$

as needed.

- **<u>Example3:</u>** Suppose

$$\boldsymbol{g}(t)=\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

and

$$\boldsymbol{x}_c=c_1\begin{pmatrix} 5\cos t \\ 2\cos t+\sin t \end{pmatrix}+c_2\begin{pmatrix} 5\sin t \\ 2\cos t-\cos t \end{pmatrix}$$

Find the correct $\boldsymbol{x}_p$ guess.

– **<u>Solution:</u>**

– <u>First guess:</u> Our first guess is based on $\boldsymbol{g}(t)$ online and since

$$\boldsymbol{g}(t)=\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}=\boldsymbol{g}(t)=\begin{pmatrix} -1 \\ 0 \end{pmatrix}\cos t+\begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t$$

then

$$\boldsymbol{x}_p=\boldsymbol{a}\cos t+\boldsymbol{b}\sin t$$

∗ But since this is part of $\boldsymbol{x}_c$ we need to make another guess.

– <u>Second guess:</u>

$$\boldsymbol{x}_p=\boldsymbol{a}t\cos t+\boldsymbol{b}t\sin t+\boldsymbol{c}\cos t+\boldsymbol{d}\sin t.$$